Large cliques or stable sets in graphs with no four-edge path and no five-edge path in the complement

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Abstract

Erdős and Hajnal [4] conjectured that, for any graph H, every graph on n vertices that does not have H as an induced subgraph contains a clique or a stable set of size $n^{\varepsilon(H)}$ for some $\varepsilon(H) > 0$. The conjecture is known to be true for graphs H with $|V(H)| \leq 4$. One of the two remaining open cases on five vertices is the case where H is a four-edge path, the other case being a cycle of length five. In this paper we prove that every graph on n vertices that does not contain a four-edge-path or the complement of a five-edge-path as an induced subgraph contains either a clique or a stable set of size at least $n^{1/6}$.

1   Introduction

All graphs in this paper are finite and simple. A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise non-adjacent vertices. For a set of vertices X, we denote by $G|X$ the subgraph of G induced by X. Let $\mathrm{Forb}(X_1, X_2, \ldots, X_k)$ be the set of all graphs G such that for all $i \in \{1, 2, \ldots, k\}$, no induced subgraph of G is isomorphic to X_i. For a vertex $v \in V(G)$, we let $N(v)$ denote the set of vertices adjacent to v and $M(v)$ the subset of vertices of $V(G)\backslash\{v\}$ that are non-adjacent to v. We say that two sets $X \subseteq V(G)$ and $Y \subseteq V(G)$ are *complete* to each other if every $x \in X$ and $y \in Y$ are adjacent. We say that X and Y are *anticomplete* to each other if every $x \in X$ and $y \in Y$ are non-adjacent. We denote by G^c the graph with vertex set $V(G)$ and edge set $\{\{u, v\} \in V(G)^2 \mid u \neq v, uv \notin E(G)\}$. We call G^c the *complement* of G.

We say that a graph H has the *Erdős-Hajnal property* if there exists $\varepsilon(H) > 0$ such that every graph on n vertices that does not have H as an induced subgraph contains either a clique or a stable set of size at least $n^{\varepsilon(H)}$. Clearly, if H has the property, then so does H^c. Erdős and Hajnal [4] conjectured that all graphs have the property. It is known to be true for every graph H with

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$|V(H)| \leq 4$. In [1], it was shown that if two graphs H_1 and H_2 have the Erdős-Hajnal property, then so does the graph constructed from H_1 by replacing a vertex $x \in V(H_1)$ by H_2 and making H_2 complete to the neighbors of x in H_1 and anticomplete to the non-neighbors of x in H_1 (this operation is known as the substitution operation). Moreover, it was shown in [3] that the triangle with two disjoint pendant edges (this graph is known as the *bull*) has the property. This leaves the four-edge-path P_4 and the cycle C_5 of length five as the remaining open cases for graphs on at most 5 vertices. This paper deals with the case where H is a four-edge path, where, in addition, we exclude the complement of a five-edge path. To be precise, we will prove that

Theorem 1.1. *Every graph* $G \in \mathrm{Forb}(P_4, P_5^c) \cup \mathrm{Forb}(P_4^c, P_5)$ *contains a clique or a stable set of size at least* $|V(G)|^{1/6}$.

For a graph G, let $\omega(G)$ denote the size of the largest clique in G and let $\chi(G)$ denote the chromatic number of G. G is called *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G. It was shown in [2] that a graph G is perfect if and only if it does not contain an odd cycle of length at least five or the complement of an odd cycle of length at least five as an induced subgraph.

We say that a function $g : V(G) \to \mathbb{R}^+$ is a *covering function for* G if $\sum_{p \in V(P)} g(p) \leq 1$ for every perfect induced subgraph P of G. For $\beta \geq 1$, we say that a graph G is β-narrow if $\sum_{v \in V(G)} g^\beta(v) \leq 1$ for every covering function g. Notice that since a graph is perfect if and only if its complement is perfect, it follows that a graph is β-narrow if and only if its complement is β-narrow. It was shown in [3] that bull-free graphs are 2-narrow. We will take a similar approach and prove that

Theorem 1.2. *All graphs in* $\mathrm{Forb}(P_4^c, P_5)$ *are 3-narrow.*

This result suffices for proving Theorem 1.1, because of the following result:

(1.3) *Let G be a β-narrow graph. Then G has a clique or stable set of size at least* $|V(G)|^{1/2\beta}$.

Proof. Let $\mathcal{P}$ be the set of all perfect induced subgraphs of G. Let $K = \max_{P \in \mathcal{P}} |V(P)|$. Consider the function $g : V(G) \to \mathbb{R}^+$ with $g(v) = 1/K$ for all $v \in V(G)$. Clearly, $\sum_{v \in V(P)} g(v) \leq 1$ for all $P \in \mathcal{P}$. Therefore, since G is β-narrow, it follows that g satisfies

$$1 \geq \sum_{v \in V(G)} g(v)^\beta = \frac{|V(G)|}{K^\beta}.$$

Equivalently, we have $K \geq |V(G)|^{\frac{1}{\beta}}$. Thus, G has a perfect induced subgraph H with $|V(H)| \geq |V(G)|^{\frac{1}{\beta}}$. Since H is a perfect graph, H satisfies $|V(H)| \leq \chi(H)\alpha(H) = \omega(H)\alpha(H)$ and hence $\max(\omega(H), \alpha(H)) \geq \sqrt{|V(H)|} \geq |V(G)|^{1/2\beta}$. Therefore, H has a clique or stable set of size at least $|V(G)|^{1/2\beta}$. Since H is an induced subgraph of G, G has a clique or stable set of size at least $|V(G)|^{1/2\beta}$. This proves (1.3). $\qquad\square$

Notice that the proof of (1.3) also shows that a graph G is 1-narrow if and only if G is perfect. Jacob Fox [5] proved that the 'converse' of (1.3) is also true:

Theorem. *Let H be a graph that has the Erdős-Hajnal property. Then, every graph in $\mathrm{Forb}(H)$ is $\beta(H)$-narrow for some $\beta(H) \geq 1$.*

This implies that the Erdős-Hajnal conjecture is equivalent to the following conjecture:

Conjecture. *For every graph H, there exists $\beta(H) \geq 1$ such that every $G \in \mathrm{Forb}(H)$ is $\beta(H)$-narrow.*

This paper is organized as follows. In Section 2, we describe tools that we will use in the rest of the paper. Section 3 deals with graphs in $\mathrm{Forb}(P_4^c, P_5)$ for which we additionally require that they have no induced copy of C_6, the cycle of length six. Finally, in Section 4 we abandon this additional requirement and finish the proof of Theorem 1.1.

2 Decompositions

We start with a number of graph decompositions and their relationship to the narrowness of graphs.

Lemma 2.1. *Let G be a graph and let $\beta \geq 1$. Suppose that for every $v \in V(G)$, either*

(i) *$G|N(v)$ is β-narrow and $G|M(v)$ is $(\beta+1)$-narrow, or*

(ii) *$G|M(v)$ is β-narrow and $G|N(v)$ is $(\beta+1)$-narrow.*

Then G is $(\beta+1)$-narrow.

Proof. Let g be a covering function for G. Choose $u \in V(G)$ with $g(u)$ maximal. We may assume that $g(u) < 1$, because every 2-vertex induced subgraph of G is perfect. Let $G_M = G|M(u)$ and $G_N = G|N(u)$. Since β-narrowness is invariant under taking complements, we may, possibly by passing to the complement, assume that G_M is $(\beta+1)$-narrow and G_N is β-narrow.

Define $f_M : V(G_M) \to \mathbb{R}^+$ by $f_M(v) = g(v)/(1-g(u))$. Let P be a perfect induced subgraph of G_M. Since $G|(V(P) \cup \{u\})$ is perfect, it follows that $\sum_{v \in V(P)} f_M(v) \leq 1$. Since G_M is $(\beta+1)$-narrow, f_M satisfies $\sum_{v \in M} f_M^{\beta+1}(v) \leq 1$ and therefore

$$\sum_{v \in M} g^{\beta+1}(v) \leq (1 - g(u))^{\beta+1}.$$

By repeating the same argument for G_N, since G_N is β-narrow, it follows that

$$\sum_{v \in N} g^\beta(v) \leq (1 - g(u))^\beta.$$

Moreover, we have, by the choice of u,

$$\sum_{v \in N} g^{\beta+1}(v) \leq g(u) \sum_{v \in N} g^\beta(v) \leq g(u)(1 - g(u))^\beta.$$

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Hence,

$$\sum_{v \in V(G)} g^{\beta+1}(v) = g^{\beta+1}(u) + \sum_{v \in M} g^{\beta+1}(v) + \sum_{v \in N} g^{\beta+1}(v)$$

$$\leq g^{\beta+1}(u) + (1 - g(u))^{\beta+1} + g(u)(1 - g(u))^{\beta}$$

$$= g^{\beta+1}(u) + (1 - g(u))^{\beta} \leq 1,$$

where the last inequality follows from the fact that the function $h(x) = x^{\beta+1} + (1 - x)^{\beta}$ is strictly convex and $h(0) = h(1) = 1$. This proves Lemma 2.1. $\square$

Let G be a graph. We say that a set $Z \subseteq V(G)$ is a *homogeneous set* in G if $1 < |Z| < |V(G)|$ and $V(G) \setminus Z = A \cup C$ where A is anticomplete to Z and C is complete to Z. In this case, we say that (Z, A, C) is a *homogeneous set decomposition* of G. It was shown in [6] that

Lemma 2.2. *Let G be a graph and let (Z, A, C) be a homogeneous set decomposition of G. Construct G' from $G|(A \cup C)$ by adding a vertex z that is complete to C and anticomplete to A. Let P_1 be a perfect induced subgraph of G' with $z \in V(P_1)$ and let P_2 be a perfect induced subgraph of $G|Z$. Then $G|(V(P_1) \cup V(P_2) \setminus \{z\})$ is perfect.*

It was shown in [3] that homogeneous set decompositions preserve β-narrowness. For our purposes, we will need a more general decomposition. We say that a set $Z \subseteq V(G)$ is a *quasi-homogeneous set* in G if there exists a partition (A, C) of $V(G) \setminus Z$ such that the following properties hold:

- $1 < |Z| < |V(G)|$.

- Either Z is complete to C or Z is anticomplete to A.

- Let G' be obtained from $G|(A \cup C)$ by adding a vertex z that is anticomplete to A and complete to C. Suppose that P_1 is a perfect induced subgraph of G' with $z \in V(P_1)$ and suppose P_2 is a perfect induced subgraph of $G|X$. Then the graph $P = G|(V(P_1) \cup V(P_2) \setminus \{z\})$ is perfect.

We say that the triple (Z, A, C) is a *quasi-homogeneous set decomposition*. In the light of Lemma 2.2, it is easy to see that a homogeneous set decomposition is a special case of a quasi-homogeneous set decomposition. Just like homogeneous set decompositions, quasi-homogeneous sets decompositions preserve β-narrowness:

Lemma 2.3. *Let G be a graph and let (Z, A, C) be a quasi-homogeneous set decomposition of G. Let H_1 be the graph obtained from $G|(A \cup C)$ by adding a vertex z anticomplete to A and complete to C and let $H_2 = G|Z$. If H_1 and H_2 are β-narrow, then G is β-narrow.*

Proof. The proof is essentially the same as the proof of **1.3** in [3], but we include it here for completeness. Let g be a covering function for G. For $i = 1, 2$, let $\mathcal{P}_i$ be the set of perfect induced subgraphs of H_i. Let $K = \max_{P \in \mathcal{P}_2} \sum_{v \in V(P)} g(v)$. Define $g_1 : V(H_1) \to \mathbb{R}^+$ as follows. For $v \in A \cup C$, let $g_1(v) = g(v)$ and let $g_1(z) = K$. Define $g_2 : V(H_2) \to \mathbb{R}^+$ by $g_2(v) = g(v)/K$ for

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$v \in V(H_2)$. From the definition of a quasi-homogeneous set decomposition, it follows that for every $P_1 \in \mathcal{P}_1$ with $z \in V(P_1)$ and every $P_2 \in \mathcal{P}_2$, $G|(V(P_1) \cup V(P_2) \setminus \{z\})$ is perfect. It follows that g_1 is a covering function for H_1. Since H_1 is β-narrow, it follows that

$$1 \geq \sum_{v \in V(H_1)} g_1^\beta(v) = \sum_{v \in A \cup C} g^\beta(v) + K^\beta.$$

Clearly, g_2 is a covering function for H_2. Thus, since H_2 is β-narrow, it follows that

$$1 \geq \sum_{v \in V(H_2)} g_2^\beta(v) = \sum_{v \in Z} \frac{g^\beta(v)}{K^\beta}.$$

Therefore,

$$\sum_{v \in Z} g^\beta(v) \leq K^\beta.$$

Finally, it follows that

$$\sum_{v \in V(G)} g^\beta(v) \leq \sum_{v \in A \cup C} g^\beta(v) + \sum_{v \in Z} g^\beta(v) \leq (1 - K^\beta) + K^\beta = 1.$$

This proves Lemma 2.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 Graphs in $\mathrm{Forb}(\mathbf{P_4^c}, \mathbf{P_5}, \mathbf{C_6})$

We start by additionally excluding the cycle of length six, C_6. Throughout the paper, we will call an induced subgraph of a graph G that is a cycle of length k a *k-gon* in G. We will often denote the vertices of a k-gon H by, for example, $h_1, h_2, \ldots, h_k$ in order. Any arithmetic involving the subscripts of these vertices is modulo k. For a k-gon H, we say that $v \in V(G) \setminus V(H)$ is a *center* for H, if v is complete to $V(H)$. Analogously, we say that v is an *anticenter* for H if v is anticomplete to $V(H)$.

We say that a graph $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ is a *composite graph* if there exist a 5-gon B in G and $a, c \in V(G) \setminus V(B)$ such that a is an anticenter for B and c is a center for B. We say that any graph in $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ is *basic* if it is not composite.

This section is organized as follows. We will first prove some basic properties of graphs in $\mathrm{Forb}(P_4^c, P_5, C_6)$. Next, we will show that composite graphs admit a quasi-homogeneous set decomposition. Finally, we will show that basic graphs satisfy the assumptions of Lemma 2.1 with $\beta = 1$. This will imply that all graphs in $\mathrm{Forb}(P_4^c, P_5, C_6)$ are 2-narrow.

3.1 Elementary properties

We will repeatedly use the following lemmas:

(3.1) *Let $G \in \mathrm{Forb}(P_4^c)$ and let f_1-f_2-f_3-f_4 be an induced path. Then no vertex is complete to $\{f_1, f_2, f_4\}$ and non-adjacent to f_3.*

Proof. Suppose for a contradiction that x is adjacent to f_1, f_2, and f_4 and not to f_3. Then x-f_3-f_1-f_4-f_2 is a four-edge antipath, a contradiction. This proves (3.1). $\qquad\square$

For a 5-gon H in a graph G, we call a vertex $x \in V(G) \setminus V(H)$ that has a neighbor in $V(H)$ an *attachment of H*. The following lemma deals with attachments of 5-gons.

(3.2) *Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and let H be a 5-gon with vertices $h_1, h_2, \ldots, h_5$ in order. Let $x \in V(G) \setminus V(H)$ with at least one neighbor in $V(H)$. Then, for some for some $i \in \{1, 2, \ldots, 5\}$, one of the following holds:*

(1) *x is complete to $V(H)$ ("center"), or*

(2) *x is adjacent to h_i and x has no other neighbor in $V(H)$ ("leaf of type i"), or*

(3) *x is adjacent to h_{i+2}, h_{i+3} and x has no other neighbor in $V(H)$ ("hat of type i"), or*

(4) *x is a adjacent to h_{i+4}, h_{i+1}, non-adjacent to h_{i+2}, h_{i+3} and the adjacency between x and h_i is arbitrary ("clone of type i").*

Proof. If x is complete to $V(H)$, then outcome (1) holds. From this and from the symmetry, we may assume that x is adjacent to h_1 and not to h_2. First, suppose that x is adjacent to h_3. From (3.1) applied first to x and h_1-h_2-h_3-h_4 and then to x and h_5-h_1-h_2-h_3, it follows that x is non-adjacent to h_4 and h_5 and thus outcome (4) holds. So we may assume that x is non-adjacent to h_3. If x is adjacent to h_4, then outcome (4) holds. So we may assume that x is non-adjacent to h_4. If x is non-adjacent to h_5, then outcome (2) holds. If x is adjacent to h_5, then outcome (3) holds. This proves (3.2). $\qquad\square$

We call an attachment x of H a *small attachment* if x is a leaf or a hat for H. Let $i \in \{1, 2, \ldots, 5\}$. We call a pair of vertices (a, b) a *pyramid of type i* for H if a and b are adjacent, a is a leaf of type i, and b is a hat of type i. We say that $\{a, b\}$ is a pyramid if (a, b) or (b, a) is a pyramid. It turns out that whenever two small attachments are adjacent, they are of the same type. The following lemma deals with combinations of small attachments:

(3.3) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and let H be a 5-gon. Suppose that u and v are small attachments of H. Then the following two statements hold:*

(a) *If u and v are adjacent, then, up to symmetry, for some $i \in \{1, 2, \ldots, 5\}$, either*

    (A1) *u and v are leaves for H of type i; or*

    (A2) *u and v are hats for H of type i; or*

    (A3) *u is a leaf for H of type i, v is a hat for H of type i, and (u, v) is a pyramid of type i for H.*

(b) *If u and v are non-adjacent, then, up to symmetry, for some $i \in \{1, 2, \ldots, 5\}$, either*

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(B1) *u is a leaf of type i and v is a leaf of type $j \in \{i-1, i, i+1\}$; or*

(B2) *u is a hat of type i and v is a hat of type $j \in \{i-2, i, i+2\}$; or*

(B3) *u is a leaf of type i and v is a hat of type $j \in \{i-2, i, i+2\}$.*

Proof. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. Since u and v are small attachments, each of u, v is either a leaf or a hat for H.

For part (a), suppose that u and v are adjacent. First assume that u is a leaf. From the symmetry, we may assume that u is a leaf of type 1 and v is either a leaf of type 1, 2 or 3, or a hat of type 1, 4 or 5. If v is a leaf of type 1, then outcome (A1) holds. If v is a hat of type 1, then outcome (A3) holds. If v is a leaf of type 2 or a hat of type 4, then u-v-h_2-h_3-h_4-h_5 is an induced five-edge path, a contradiction. If v is a leaf of type 3 or a hat of type 5, then u-v-h_3-h_4-h_5-h_1-u is an induced cycle of length six, a contradiction. This finishes the case when u is a leaf. So we may now assume that both u and v are hats. From the symmetry, we may assume that u is a hat of type 1 and v is a hat of type 1, 2 or 3. If v is a hat of type 1, then outcome (A2) holds. If v is a hat of type 2, then u-v-h_5-h_1-h_2-h_3-u is an induced cycle of length six, a contradiction. If v is a hat of type 3, then the adjacencies of v with respect to the path u-h_4-h_5-h_1 contradict (3.1). This proves part (a).

For part (b), suppose that u and v are non-adjacent. First assume that u is a leaf. From the symmetry, we may assume that u is of type 1 and v is either a leaf of type 1, 2 or 3, or a hat of type 1, 4, 5. If v is a leaf of type 1 or 2, then (B1) holds. If v is a leaf of type 3 or a hat of type 5, then u-h_1-h_5-h_4-h_3-v is an induced five-edge path, a contradiction. If v is a hat of type 1 or 4, then outcome (B3) holds. This finishes the case when u is a leaf. We may therefore assume that u and v are both hats for H. From the symmetry, we may assume that u is a hat of type 1 and v is a hat of type 1, 2 or 3. If v is a hat of type 1 or 3, then (B2) holds. If v is a hat of type 2, then u-h_3-h_2-h_1-h_5-v is an induced five-edge path, a contradiction. This proves part (b), thereby completing the proof of (3.3). $\square$

(3.4) *Let $G \in \mathrm{Forb}(P_4^c, P_5)$. Let H be a 5-gon in G and suppose that x is a small attachment of H. Then, every neighbor $y \in V(G) \setminus V(H)$ of x is an attachment of H.*

Proof. Suppose that $y \in V(G) \setminus V(H)$ is adjacent to x but y has no neighbor in $V(H)$. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. We may assume that x is adjacent to h_1 and anticomplete to $\{h_2, h_3, h_4\}$. Now y-x-h_1-h_2-h_3-h_4 is an induced five-edge path, a contradiction. This proves (3.4). $\square$

(3.5) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and let H be a 5-gon. Let (a, b) and (a', b') be two disjoint pyramids for H. Then (a, b) and (a', b') are pyramids of the same type.*

Proof. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. From the symmetry, we may assume that (a, b) is a pyramid of type 1 and (a', b') is a pyramid of type 1, 2 or 3. If (a', b') is of type 1, then the claim holds. If (a', b') is a pyramid of type 2, then b is a hat of type 1 for H and b' is a hat of

type 2 for H, contrary to (A2) and (B2) of (3.3). If (a', b') is a pyramid of type 3, then a is a leaf of type 1 and a' is a leaf of type 3 for H, contrary to (A1) and (B1) of (3.3). This proves (3.5). $\square$

3.2 Composite graphs

Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ be a graph. Our goal is to produce a quasi-homogeneous set. In order to do so, we need to understand how different 5-gons interact with each other. To this end, we consider the following auxiliary graph. Let B be a 5-gon in G and let $\mathcal{W}$ be a graph with the follows properties:

(a) The vertices of $\mathcal{W}$ are 5-gons in G, and B is a vertex of $\mathcal{W}$.

(b) Two 5-gons H and H' are adjacent if and only if one of the following holds:

    (b1) $|V(H) \cap V(H')| = 4$ and $x \in V(H) \setminus V(H')$ is a clone for H'. In this case, we say that H and H' are *clone neighbors* and we call the edge HH' a *clone edge*.

    (b2) $B \in \{H, H'\}$, $|V(H) \cap V(H')| = 3$ and $\{x, y\} = V(H) \setminus V(H')$ is a pyramid for H'. In this case, we say that H and H' are *pyramid neighbors* and we call the edge HH' a *pyramid edge*.

(c) $\mathcal{W}$ is connected.

We call such a graph $\mathcal{W}$ a C_5-*structure around B in G*. Note that we do not require that *all* 5-gons in G are vertices of $\mathcal{W}$. Also note that the adjacency of two 5-gons is well-defined because property (b) is symmetric. We say that a C_5-structure $\mathcal{W}$ is *maximal* if $|V(\mathcal{W})|$ is maximal and, subject to that, $|E(\mathcal{W})|$ is maximal. Let $U(\mathcal{W}) = \bigcup_{H \in V(\mathcal{W})} V(H)$ denote the set of vertices of G that are 'covered' by $\mathcal{W}$.

Let $H \in V(\mathcal{W})$ and let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. Let $i \in \{1, 2, \ldots, 5\}$ and let x be a clone of type i for H. We will write $H/x = G|((V(H) \setminus \{h_1\}) \cup \{x\})$ and we will say that H/x is *obtained from H by cloning h_i and x is a clone in the position of h_i*. For two 5-gons $F, H \in V(\mathcal{W})$, let $\mathrm{dist}(F, H)$ be the number of edges in a shortest path from F to H in $\mathcal{W}$.

Let us first prove a number of claims about C_5-structures:

(3.6) *Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and let B be a 5-gon in G. Let $\mathcal{W}$ be a C_5-structure around B. Suppose that $H \in V(\mathcal{W})$ and $H' \in V(\mathcal{W})$ are clone neighbors. If c is a center for H, then c also a center for H'.*

Proof. Let c be a center for H. From the definition of a clone edge, it follows that $|V(H) \cap V(H')| = 4$. Since c is complete to $V(H)$, it follows that c has at least four neighbors in $V(H')$. Therefore, it follows from (3.2) that c is complete to $V(H')$. This proves (3.6). $\square$

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(3.7) *Let $G \in \text{Forb}(P_4^c, P_5)$ and let B be a 5-gon in G. Let $\mathcal{W}$ be a maximal C_5-structure around B. Let c be a center for some 5-gon in $V(\mathcal{W})$. Then either c is a center for every $H \in V(\mathcal{W})$ or $c \in U(\mathcal{W})$.*

Proof. If c is complete to all $H \in V(\mathcal{W})$, then the claim holds. So we may assume that c is not complete to at least one 5-gon in $V(\mathcal{W})$. Let $H_1, H_2 \in V(\mathcal{W})$ be such that c is complete to H_1 but not to H_2 and, subject to that, such that $\text{dist}(H_1, H_2)$ is minimum. Clearly, since c is complete to $V(H_1)$ and not to $V(H_2)$, it follows that $H_1 \neq H_2$. Since $\text{dist}(H_1, H_2)$ is minimum, it follows that H_1 and H_2 are neighbors. It follows from (3.6) that H_1 and H_2 are pyramid neighbors. We may write $H_1 = h_1\text{-}h_2\text{-}h_3\text{-}h_4\text{-}h_5\text{-}h_1$ and $H_2 = h_1\text{-}a\text{-}b\text{-}h_4\text{-}h_5\text{-}h_1$. Since c is complete to $V(H_1)$, it follows that c has at least three neighbors in $V(H_2)$. Hence, since c is not complete to $V(H_2)$, it follows from (3.2) that c is a clone for H_2. Therefore, H_2/c is a 5-gon. From the maximality of $\mathcal{W}$, it follows that $H_2/c \in V(\mathcal{W})$ and, thus, that $c \in U(\mathcal{W})$. This proves (3.7). $\qquad\square$

(3.8) *Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ and let B be a 5-gon in G. Let $\mathcal{W}$ be a maximal C_5-structure around B. Suppose that $H \in V(\mathcal{W})$ and $H' \in V(\mathcal{W})$ are clone neighbors and let x be such that $H' = H/x$. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. Let $i \in \{1, 2, \ldots, 5\}$ and suppose that (p, q) is a pyramid of type i for H. Then either*

(1) *(p, q) is also a pyramid of type i for H', or*

(2) *x is a clone of type $j \in \{i-1, i+1\}$ for H and x is complete to $\{p, q, h_j\}$.*

Proof. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. From the symmetry, we may assume that (p, q) is a pyramid of type 1 for H and x is a clone of type 1, 2 or 3 for H. First assume that x is a clone of type 1 for H. It follows from (3.2) applied to q and H' that x is not adjacent to q. Therefore, q is a hat for H'. Since p is a neighbor of q, it follows from (3.4) that p has a neighbor in $V(H')$. It follows that p is adjacent to x. Thus, (p, q) is a pyramid for H' and outcome (1) holds. Next, assume that x is a clone of type 2 for H. Then it follows from (3.2) applied to x and H' that x is either complete or anticomplete to $\{p, q\}$. If x is anticomplete to $\{p, q\}$, then (p, q) is a pyramid for H' and thus outcome (1) holds. If x is complete to $\{p, q\}$, then it follows from (3.3) that x is adjacent to h_2. Hence, outcome (2) holds. So we may assume that x is a clone of type 3 for H. First suppose that p is adjacent to x. From (3.2) applied to x and the 5-gon $h_1\text{-}h_2\text{-}h_3\text{-}q\text{-}p\text{-}h_1$, it follows that x is anticomplete to $\{q, h_3\}$. But now the adjacencies of q with respect to $h_3\text{-}h_4\text{-}x\text{-}p$ contradict (3.1). This proves that p is non-adjacent to x. But now, since p is a leaf of type 1 for H', q is a small attachment of H', and p and q are adjacent, it follows from (3.3) that q is a hat of type 1 for H' and (p, q) is a pyramid for H'. Hence, outcome (1) holds. This proves (3.8). $\qquad\square$

The goal in this section is to prove the following:

(3.9) *Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ be a composite graph. Let B be a 5-gon in G and let A and C be the set of vertices that are complete and anticomplete, respectively, to $V(B)$. Let $\mathcal{W}$ be a maximal C_5-*

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structure around B. Then $(U(\mathcal{W}), A \setminus U(\mathcal{W}), C \setminus U(\mathcal{W}))$ is a quasi-homogeneous set decomposition of G.

As a first step in this direction, we prove the following lemma which states that $U(\mathcal{W})$ does not contain both all centers and all anticenters of B. This is useful, because in order for $U(\mathcal{W})$ to be a quasi-homogeneous set, we should have $|U(\mathcal{W})| < |V(G)|$.

(3.10) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and let B be a 5-gon with both a center and an anticenter. Let $\mathcal{W}$ be a maximal C_5-structure around B. Then $V(G) \setminus U(\mathcal{W}) \neq \emptyset$.*

Proof. We may assume that all centers and all anticenters for B are contained in $U(\mathcal{W})$.

(i) *B and every pyramid neighbor of B in $\mathcal{W}$ has a pyramid.*

We first claim that B has a pyramid. For suppose not. Let x be a center for B. Then it follows from (3.6) that x is a center for all $H \in V(\mathcal{W})$. In particular, for every $H \in V(\mathcal{W})$, $x \notin V(H)$. Therefore, $x \notin U(\mathcal{W})$, contrary to our assumption. Now let B' be any pyramid neighbor of B. Clearly, $\{p, q\} = V(B) \setminus V(B')$ is a pyramid for B'. This proves **(i)**.

Now let a be an anticenter for B. We first show that:

(ii) *a is anticomplete to every pyramid (p, q) for B and a is an anticenter for every pyramid neighbor of B in $\mathcal{W}$.*

Let (p, q) be a pyramid for B. Suppose that $z \in \{p, q\}$ is adjacent to a. Since z is a small attachment of B, it follows from (3.4) that a has a neighbor in $V(B)$, contrary to the assumption that a is an anticenter for B. Since every pyramid neighbor H of B satisfies $V(H) \subseteq (V(B) \cup \{p', q'\})$ for some pyramid $\{p', q'\}$ for B, it follows from the above that a is an anticenter for every pyramid neighbor of B. This proves **(ii)**.

Since $a \in U(\mathcal{W})$ there exists a 5-gon $H^* \in V(\mathcal{W})$ such that $a \in V(H^*)$ and, subject to that, such that $\mathrm{dist}(B, H^*)$ is minimum. Let P be a shortest path from H^* to B in $\mathcal{W}$ and write $P = H^*\text{-}H^1\text{-}H^2\text{-}\cdots\text{-}H^k$, where $H^k = B$ and $k = \mathrm{dist}(B, H^*)$. From the definition of a C_5-structure, it follows that all edges in P are clone edges, except possibly $H^{k-1}\text{-}H^k$.

(iii) *$H^* = H^1/a$, $k \geq 2$, and H^1 is not a pyramid-neighbor of B.*

First suppose that $H^1 = B$. If H^* and B are pyramid neighbors, then it follows from **(ii)** that a is anticomplete to H^*, a contradiction. If H^* and B are clone neighbors, then, since $|V(B) \cap V(H^*)| = 4$ and a has two neighbors in $V(H^*)$, it follows that a has at least one neighbor in B, contradicting the fact that a is an anticenter for B. This proves that $H^1 \neq B$ and, thus, that $k \geq 2$. It follows from the definition of $\mathcal{W}$ that $H^*\text{-}H^1$ is a clone edge. Since $a \in V(H^*)$ and $a \notin V(H^1)$, it follows that $H^* = H^1/a$. Since a has a neighbor in $V(H^1)$, it follows from **(ii)** that H^1 is not a pyramid neighbor of B. This proves **(iii)**.
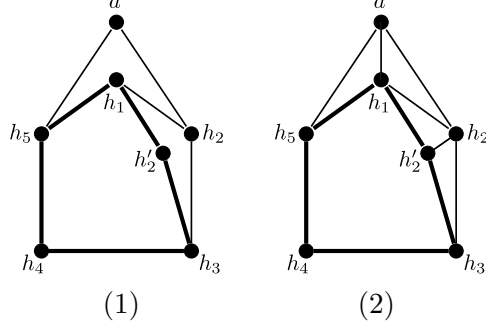
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Figure 1: The outcomes of **(v)**.

(iv) *a is not a clone for H^i for $i \geq 2$.*

Suppose that a is a clone for H^i. Then H^i/a-H^i-H^{i+1}-$\cdots$-H^k is a path between B and a 5-gon containing a that is shorter than P, contradicting the choice of H^*. This proves **(iv)**.

Let $h_1, h_2, \ldots, h_5$ be the vertices of H^1 in order. From the symmetry, we may assume that a is adjacent to h_2 and h_5, and possibly to h_1. Let us now consider H^2.

(v) *Up to symmetry, H^2 is obtained from H^1 by cloning h_2. Let h'_2 be such that $H^2 = H^1/h'_2$. Then h_2 is non-adjacent to a, and either (see Figure 1)*

(1) *ah_1 and $h_2h'_2$ are either both non-edges, or*

(2) *ah_1 and $h_2h'_2$ are either both edges.*

Moreover, $k \geq 3$ and H^2 is not a pyramid neighbor of B.

It follows from **(iii)** that H^1 and H^2 are clone neighbors. From the symmetry, we may assume that H^2 is obtained from H^1 by cloning h_1, h_2, or h_3. It follows from **(iv)** that H^2 is not obtained from H^1 by cloning h_1. Suppose next that H^2 is obtained from H^1 by cloning h_3. Let h'_3 be such that $H^2 = H^1/h_3$. It follows from (3.2) that a is a clone for H^2, contradicting **(iv)**. Therefore, we may assume that H^2 is obtained from H^1 by cloning h_2. Let h'_2 be such that $H^2 = H^1/h'_2$. Because, from **(iv)**, a is not a clone for H^2, it follows that h'_2 is non-adjacent to a. If h'_2 is adjacent to h_2 and h_1 is non-adjacent to a, then h_2-h_5-h'_2-a-h_1 is an induced four-edge antipath, a contradiction. Likewise, if h'_2 is non-adjacent to h_2 and h_1 is adjacent to a, then h_1-h_3-a-h'_2-h_2 is a four-edge antipath, a contradiction. This proves that ah_1 and $h_2h'_2$ are either both edges or both non-edges.

Since a has a neighbor in H^2, it follows that $H^2 \neq B$ and hence that $k \geq 3$. Using **(ii)**, it follows that H^2 is not a pyramid neighbor of B. This proves **(v)**.

Let H^2 and h'_2 be as in **(v)**. It follows from **(v)** that we may now consider H^3, $H^3 \neq B$ and H^3 is not a pyramid neighbor of B. Therefore, H^2 and H^3 are clone neighbors.

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(vi) *Up to symmetry, H^3 is either (see Figure 2)*

(1) *obtained from H^2 by cloning h_5, $h_5' \in V(H^3) \setminus V(H^2)$ is anticomplete to $\{a, h_2, h_5\}$, and $ah_1, h_2'h_2$ are non-edges; or*

(2) *obtained from H^2 by cloning h_5, $h_5' \in V(H^3) \setminus V(H^2)$ is adjacent to h_5 and anticomplete to $\{a, h_2\}$, and $ah_1, h_2'h_2$ are edges, or*

(3) *obtained from H^2 by cloning h_1, $h_1' \in V(H^3) \setminus V(H^2)$ is adjacent to h_1 and anticomplete to $\{a, h_2\}$, and $ah_1, h_2'h_2$ are edges.*

Moreover, $k \geq 4$ and H^3 is not a pyramid neighbor of B.

Since H^2 and H^3 are clone neighbors, we may assume that H^2 is obtained from H^1 by cloning h_2. It follows from **(v)** that h_2' is non-adjacent to a. H^3 is not obtained from H^2 by cloning h_2', because if it is, then H^3 is adjacent to H^1, contrary to the minimality of P.

Also note that H^3 has no neighbor $H' \in V(\mathcal{W})$ such that a is a clone for H'. Because if so, then H'/a-H'-H^3-H^4-$\cdots$-H^k is a path between B and a 5-gon containing a that is shorter than P, a contradiction.

There are four cases to consider:

(a) $\underline{H^3 \text{ is obtained from } H^2 \text{ by cloning } h_1.}$ (see Figure 3.a.) Let h_1' be such that $H^3 = H^2/h_1'$. If h_1' is adjacent to h_2, then H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and a is a clone for H^3/h_2, a contradiction. Therefore, h_1' is non-adjacent to h_2. First suppose that H^2 satisfies outcome (1) of **(v)**. Since h_1-h_3-h_1'-h_2-h_2' is not an induced four-edge antipath, it follows that h_1' is non-adjacent to h_1. If h_1' is non-adjacent to a, then a and h_2 are adjacent leaves of different types for H^3, contrary to (3.3). Therefore, h_1' is adjacent to a. But now h_1'-h_1-a-h_2'-h_5 is an induced four-edge antipath, a contradiction. Next suppose that H^2 satisfies outcome (2) of **(v)**. From the fact that a-h_2'-h_5-h_2-h_1' is not an induced four-edge antipath, it follows that a is non-adjacent to h_1'. It follows, from the fact that h_2'-h_5-h_2-h_1'-h_1 is not a four-edge antipath, that h_1' is adjacent to h_1. Hence, outcome (3) holds.
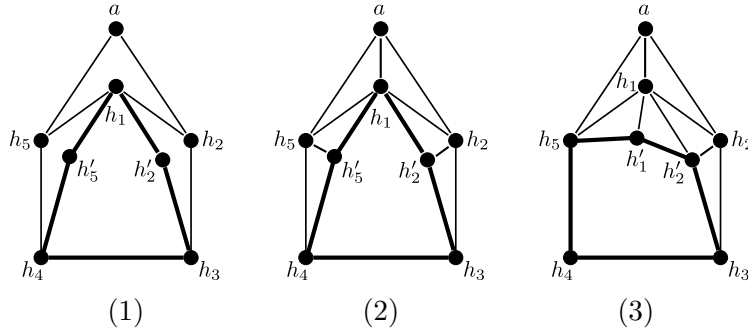

Figure 2: The outcomes of **(vi)**.

(b) H^3 is obtained from H^2 by cloning h_3. (see Figure 3.b.) Let h_3' be such that $H^3 = H^2/h_3'$. Suppose that h_3' is adjacent to a. Then, it follows from (3.2) that a is a clone for H^3, contrary to **(iv)**. Hence, h_3' is non-adjacent to a. It follows that a is either a leaf of type 5 or a hat of type 3 for H^3. If h_3' is adjacent to h_2, then H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and a is a clone for H^3/h_3, a contradiction. Therefore, h_3' is non-adjacent to h_2 and hence h_2 is a leaf of type 1 for H^3. But now a and h_2 are small attachments of H^3 but they have different types, contrary to (3.3).

(c) H^3 is obtained from H^2 by cloning h_4. (see Figure 3.c.) Let h_4' be such that $H^3 = H^2/h_4'$. From (3.2) applied to h_2 and H^3, it follows that h_4' is non-adjacent to h_2 and, in particular, that h_2 is a clone for H^3. But now H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and a is a clone for H^3/h_2, a contradiction.

(d) H^3 is obtained from H^2 by cloning h_5. (see Figure 3.d.) Let h_5' be such that $H^3 = H^2/h_5'$. From (3.2) applied to h_2 and H^3, it follows that h_5' is non-adjacent to h_2 and, in particular, that h_2 is a clone for H^3. Since a is not a clone for H^3/h_2, it follows that a is non-adjacent to h_5'. If H^2 satisfies outcome (1) of **(v)**, then because h_1-a-h_5'-h_2-h_5 is not an induced four-edge antipath, it follows that h_5 is non-adjacent to h_5' and hence outcome (1) holds. If H^2 satisfies outcome (2) of **(v)**, then since h_5-h_5'-a-h_4-h_1 is not an induced four-edge antipath, it follows that h_5 is adjacent to h_5', and hence outcome (2) holds.

Now suppose that $H^3 = B$ or H^3 is a pyramid neighbor of B. Since a is an anticenter for B and for every pyramid neighbor of B, it follows that H^3 satisfies outcome (1). It follows from **(i)** and **(ii)** that H^3 has a pyramid (p, q) that is anticomplete to a. From the symmetry, we may assume that (p, q) is a pyramid of type 1, 2, or 3. First suppose that (p, q) is a pyramid of type 1 for H^3. It follows from (3.8) that $\{p, q\}$ is anticomplete to $\{h_2, h_5\}$. But now h_2 is a leaf for the 5-gon $F = h_1$-p-q-h_4-h_5'-h_1, a is adjacent to h_2 and a has no neighbor in F, contrary to (3.4). Next suppose that (p, q) is a pyramid of type 2 for H^3. Then it follows from (3.8) that p is non-adjacent to h_5. Hence, a is a leaf of type 5 and p is a leaf of type 2 for H^3/h_5, contrary to (3.3). So we may assume that (p, q) is a pyramid of type 3 for H^3. It follows from (3.8) that
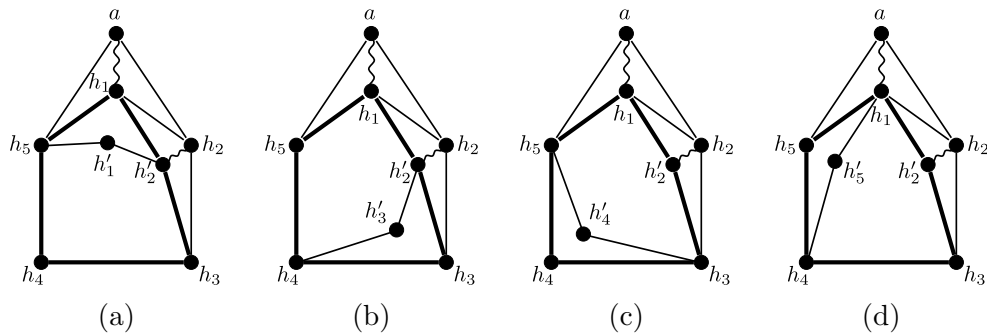

Figure 3: Potential neighbors of H^2 if H^2 satisfies (1) of **(vi)**.
The "wiggly" edges represent arbitrary adjacencies.

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p is non-adjacent to h_5. Hence, a is a leaf of type 5 and p is a leaf of type 3 for H^3/h_5, contrary to (3.3). This proves that H^3 is not B or a pyramid neighbor of B and therefore that $k \geq 4$. This proves **(vi)**.

Let H^3 be as in **(vi)**. It follows from **(vi)** that we may now consider H^4, which is a clone neighbor of H^3. Now, again, since P is a shortest path from a 5-gon that contains a to B, it follows that there is no one- or two-edge path in $\mathcal{W}$ from H^4 to a 5-gon for which a is clone.

First, suppose that H^3 satisfies outcome (1) or (2) of **(vi)**. Let h_5' be as in outcome (1) and (2) of **(vi)**. From the symmetry, we may assume that H^4 is obtained from H^3 by cloning h_1, h_2', or h_3. We need to check a number of cases:

(a) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_1}$. Let h_1' be such that $H^4 = H^3/h_1'$. First suppose that h_1' is non-adjacent to h_2. It follows that h_2 is a leaf of type 3 or a hat of type 5 for H^4. Since a is adjacent to h_2, it follows from (3.4) that a is adjacent to h_1'. But now a is a leaf of type 1 for H^4 and a is adjacent to h_2, contrary to (3.3). Therefore, h_1' is adjacent to h_2 and, from the symmetry, h_1' is adjacent to h_5. But now the path H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(b) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_2'}$. Now H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(c) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_3}$. Let h_3' be such that $H^4 = H^3/h_3'$. From (3.2) applied to h_5 and H^4, it follows that h_3' is non-adjacent to h_5 and, in particular, that h_5 is a clone for H^4. Since a is a not a clone for H^4/h_5, it follows from (3.2) that a is non-adjacent to h_3'. If h_3' is adjacent to h_2, then H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is a clone, a contradiction. Hence, h_3' is non-adjacent to h_2 and therefore h_2 is a small attachment of H^4. Since a is adjacent to h_2, it follows from (3.4) that a is adjacent to h_1 and hence that outcome (2) of **(vi)** holds. But now a is a leaf of type 1 for H^4, h_2 is a hat of type 4 for H^4, and a and h_2 are adjacent, contrary to (3.3).

This proves that H^3 does not satisfy outcome (1) or outcome (2) of **(vi)**. So next suppose that H^3 satisfies outcome (3) of **(vi)**. We need to check a number of cases:

(a) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_1'}$. H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(b) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_2'}$. Let h_2'' be such that $H^4 = H^3/h_2''$. Since a is not a clone for H^4, it follows that h_2'' is non-adjacent to a. If h_2'' is adjacent to h_1, then H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction. Therefore h_2'' is non-adjacent to h_1. But now h_1 is is a hat of type 3 and a is a leaf of type 5 for H^4, and h_1 and a are adjacent, contrary to (3.4).

(c) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_3}$. Let h_3' be such that $H^4 = H^3/h_3'$. It follows from (3.2) that a is non-adjacent to h_3'. From (3.2) applied to h_1 and H^4, it follows that h_3' is

non-adjacent to h_1. If h_3' is non-adjacent to h_2, then h_2 and a are leaves of type 3 and 5, respectively, and a and h_2 are adjacent, contrary to (3.3). Therefore, h_3' is adjacent to h_2. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(d) H^4 is obtained from H^3 by cloning h_4. Let h_4' be such that $H^4 = H^3/h_4'$. By (3.2) applied to h_1 and H^4, it follows that h_4' is non-adjacent to h_1. By (3.2) applied to h_2 and H^4/h_1, it follows that h_4' is non-adjacent to h_2. By (3.2) applied to a and $H^4/h_1/h_2$, it follows that h_4' is non-adjacent to a. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(e) H^4 is obtained from H^3 by cloning h_5. Let h_5' be such that $H^4 = H^3/h_5'$. From (3.2) applied to h_2 and H^4, it follows that h_5' is non-adjacent to h_2. If h_5' is non-adjacent to h_1, then h_1 and h_2 are hats of type 4 and 5, respectively, and h_1 and h_2 are adjacent, contrary to (3.3). Therefore, h_5' is adjacent to h_1. Since h_2 is a hat for H^4 and a is adjacent to h_2, it follows from (3.4) that a is adjacent to h_5'. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

This proves that H^3 does not satisfy any of the outcomes of **(vi)**, a contradiction. This completes the proof of (3.10). $\square$

Next, we are interested in how vertices in $V(G)\backslash U(\mathcal{W})$ can attach to $U(\mathcal{W})$ where $\mathcal{W}$ is a C_5-structure.

(3.11) *Let* $G \in \text{Forb}(P_4^c, P_5, C_6)$ *and let* B *be a 5-gon. Let* $\mathcal{W}$ *be a maximal* C_5-structure around B. *Let* $x \in V(G) \setminus U(\mathcal{W})$ *and assume that* x *is not a center for* $\mathcal{W}$. *Let* u *and* v *be two non-adjacent neighbors of* x *and assume that* $u \in U(\mathcal{W})$. *Then, for every* $H \in V(\mathcal{W})$ *such that* $u \in V(H)$, v *is a clone for* H *in the same position as* u *and, in particular,* $v \in U(\mathcal{W})$.

Proof.

(i) *If* (a, b) *is a pyramid for some* $H \in V(\mathcal{W})$, *then* $\{a, b\} \subset U(\mathcal{W})$.

Let H^* be a 5-gon for which (a, b) is a pyramid and, subject to that, such that $\text{dist}(H^*, B)$ is minimum. Let $h_1, h_2, \ldots, h_5$ be the vertices of H^* in order. From the symmetry, we may assume that (a, b) is a pyramid of type 1 for H^*.

Let P be a shortest path from H^* to B. It follows from the definition of a maximal C_5-structure that, if $H^* = B$, then $\{a, b\} \subset U(\mathcal{W})$. So we may assume that $H^* \neq B$ and hence that $|E(P)| \geq 1$. Let H^1 be the neighbor of H^* in P. Since H^* was chosen with $\text{dist}(H^*, B)$ minimum, it follows that $\{a, b\}$ is not a pyramid for H^1.

First suppose that H^1 is a clone neighbor of H^*. Let x be such that $H^1 = H^*/x$. From (3.8) and the fact that $\{a, b\}$ is not a pyramid for H^1, it follows that H^1 is obtained from H^* by cloning h_2 or h_5 and x is complete to $\{a, b\}$. But now, from the maximality of $\mathcal{W}$, H^1-H^1/b-$H^1/b/a$ is

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a path in $\mathcal{W}$ and hence $\{a, b\} \subset U(\mathcal{W})$.

Therefore, we may assume that H^1 is a pyramid neighbor of H^*. From the definition of a maximal C_5-structure and the fact that $H^* \neq B$, it follows that $H^1 = B$. Let $\{p, q\} = V(B) \setminus V(H^*)$. We claim that either (p, q) or (q, p) is a pyramid of type 1. If $\{p, q\} \cap \{a, b\} = \emptyset$, then, since (a, b) is a pyramid of type 1 for H^*, it follows from (3.5) that (p, q) or (q, p) is a pyramid of type 1 for H^*. If $\{p, q\} \cap \{a, b\} \neq \emptyset$, then it follows from the definition of a pyramid that (p, q) or (q, p) is a pyramid of type 1 for H^*. Hence, we may assume that $V(H^1) = V(B) = \{h_1, p, q, h_4, h_5\}$. This proves that (p, q) or (q, p) is a pyramid of type 1. From the symmetry, we may assume that (p, q) is a pyramid of type 1.

If $(a, b) = (p, q)$, then $\{a, b\} \subset V(B)$ and hence $\{a, b\} \subset U(\mathcal{W})$. If $a \neq p$ and $b = q$, then a is a clone for B and $b \in V(B)$ and, therefore, $\{a, b\} \subset U(\mathcal{W})$. If $a = p$ and $b \neq q$, then b is a clone for B and $a \in V(B)$ and, therefore, $\{a, b\} \subset U(\mathcal{W})$.

So we may assume that $\{a, b\} \cap \{p, q\} = \emptyset$. Now first suppose that a is adjacent to q. Then a is a clone for B and b is a clone for B/a. Hence, by maximality of $\mathcal{W}$, it follows that $B/a, B/a/b \in V(\mathcal{W})$ and, therefore, that $\{a, b\} \subset U(\mathcal{W})$. Next, suppose that b is adjacent to p. Then b is a clone for B and a is a clone for B/b. Hence, by maximality of $\mathcal{W}$, it follows that $B/b, B/b/a \in V(\mathcal{W})$ and, therefore, that $\{a, b\} \subset U(\mathcal{W})$.

It follows that we may assume that the only possible edges between $\{a, b\}$ and $\{p, q\}$ are ap and bq. It follows from (3.3) that exactly one of ab and pq is an edge and hence that $\{a, b\}$ is a pyramid for B. If a is adjacent to p, then (b, a) is a pyramid of type 4 for B, contrary to (3.5). If b is adjacent to q, then (a, b) is a pyramid of type 1 for B. By maximality of $\mathcal{W}$, it follows that $\{a, b\} \subset U(\mathcal{W})$. This proves **(i)**.

Let $H \in V(\mathcal{W})$ such that $u \in V(H)$ and let h_1, h_2, h_3, h_4, h_5 be the vertices of H in order. From the symmetry, we may assume that $h_1 = u$. It follows from (3.7) and the assumption that x is not a center for $\mathcal{W}$ that x is not complete to $V(H)$. Moreover, since $\mathcal{W}$ is maximal and $x \notin U(\mathcal{W})$, it follows that x is not a clone for H. Therefore x is either a leaf or a hat for H. From the symmetry, we may assume that x is anticomplete to $\{h_2, h_3, h_4\}$, but possibly adjacent to h_5. Because x is a small attachment of H and u is adjacent to x, it follows from (3.4) that u has at least one neighbor in $V(H)$. Since u and v are non-adjacent, v is not complete to H. Hence, it follows from (3.2) that v is either a small attachment or a clone for H.

First suppose that v is a small attachment of H. Then, from (3.3) and the fact that u and v are non-adjacent, it follows that (x, v) is a pyramid for H. But now, by **(i)**, $\{x, v\} \subset U(\mathcal{W})$, contradicting the fact that $x \notin U(\mathcal{W})$.

So we may assume that v is a clone for H. If v is adjacent to h_2 and h_5, then the claim holds. Therefore, we may assume that v is adjacent to at most one of h_2, h_5. Since u and v are non-adjacent, it follows that v is a clone of type 3 or 4. If v is a clone of type 3, then it follows from (3.2) that x is a clone for H/v and hence $x \in U(\mathcal{W})$, a contradiction. If v is a clone type 4, then again x is a clone for H/v and hence $x \in U(\mathcal{W})$, a contradiction. This proves (3.11). $\qquad\square$

We are now in a position to prove (3.9).

Proof of (3.9). Let B be a 5-gon with a center and an anticenter and let $\mathcal{W}$ be a maximal C_5-structure around B. Let $Z = U(\mathcal{W})$, let C be the set of centers for $\mathcal{W}$ and let A be $V(G) \setminus (Z \cup C)$. It follows from (3.10) that $A \cup C \neq \emptyset$. We claim that (Z, A, C) is a quasi-homogeneous set decomposition of G. Clearly, C is complete to Z. Let P be as in the definition of a quasi-homogeneous set decomposition. Suppose that P is not perfect. Since P is an induced subgraph of G, it does not have an induced four-edge antipath or an induced five-edge path. It follows that P contains an induced cycle F of length five. Let $f_1, f_2, \ldots, f_5$ be the vertices of F in order.

(i) *No edge of F has one endpoint in Z and one endpoint in C.*

From the symmetry, we may assume that $f_1 \in Z$ and $f_2 \in C$. Since C is complete to Z, and f_4 is non-adjacent to f_1 and f_2, it follows that $f_4 \in A$. Moreover, since f_5 is non-adjacent to f_2, it follows for the same reason that $f_5 \in A \cup C$. If $f_5 \in A$, then (3.11) with $x = f_5$, $u = f_1$ and $v = f_4$, implies that $f_4 \in Z$, a contradiction. Therefore, we may assume that $f_5 \in C$. Because f_3 is non-adjacent to f_1 and f_5, it follows that $f_3 \notin C \cup Z$, and hence that $f_3 \in A$. But now z-f_2-f_3-f_4-f_5-z is an induced cycle of length five in P_1, contradicting the fact that P_1 is perfect. This proves **(i)**.

Let P^* be obtained from P by deleting all edges between A and Z. It follows from Lemma 2.2 that P^* is perfect. Therefore, F is not an induced subgraph of P^*. It follows that there exist two adjacent vertices $a \in Z$ and $b \in A$ such that $a, b \in V(F)$, say $f_1 = a$ and $f_2 = b$.

Let $H \in V(\mathcal{W})$ be such that $f_1 \in V(H)$. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. We may assume that $f_1 = h_1$.

(ii) *No vertex $w \in A$ is a clone or a center for H.*

If w is a clone for H, then it follows from the maximality of $\mathcal{W}$ that $w \in Z$, a contradiction. If w is a center for H, then it follows from (3.7) that $w \in Z \cup C$, a contradiction. This proves **(ii)**.

(iii) *f_3 is a clone of type 1 for H and $\{f_3, f_4, f_5\} \subset Z$.*

Since f_1 is non-adjacent to f_3, it follows from (3.11) that $f_3 \in Z$ and f_3 is a clone in the same position as f_1 for H. It follows from **(i)** that $f_5 \in A \cup Z$. Suppose that $f_5 \in A$. Since f_4 is non-adjacent to f_1, it follows from (3.11) that f_4 is also a clone of type 1 for H. If f_5 is adjacent to both h_5 and h_2, then it follows from (3.2) that f_5 is a clone or a center for H, contrary to **(ii)**. Therefore, from the symmetry, we may assume that f_5 is non-adjacent to h_2. But now h_2-f_5-f_3-f_1-f_4 is an induced four-edge antipath, a contradiction. This proves that $f_5 \in Z$ and, from the symmetry, that $f_4 \in Z$, and hence this proves **(iii)**.

Since f_5 is adjacent to f_1, but not to f_3, it follows that $f_5 \notin V(H)$. Since f_4 is adjacent to f_3 but not to f_1, it follows that $f_4 \notin V(H)$. It follows from **(ii)** that f_2 is not a clone or a center for H and hence that f_2 is non-adjacent to h_3 and h_4.

We claim that $\{f_4, f_5\}$ anticomplete to $\{h_2, h_5\}$. For suppose not. From the symmetry, we may assume that f_4 is adjacent to h_2. If f_4 is non-adjacent to h_5, then f_3-f_1-f_4-h_5-h_2 is an induced four-edge antipath, a contradiction. Therefore, f_4 is adjacent to h_5. If f_2 is adjacent to both h_2 and h_5, then it follows from (3.2) that f_2 is a clone or a center for H, contrary to **(ii)**. Hence, from the symmetry, we may assume that f_2 is non-adjacent to h_2. But now f_3-f_1-f_4-f_2-h_2 is an induced four-edge antipath, a contradiction. This proves that $\{f_4, f_5\}$ anticomplete to $\{h_2, h_5\}$.

It follows from (3.4) applied to h_3, h_4 and h_2-f_3-f_4-f_5-f_1-h_2 that there is at least one edge between $\{h_3, h_4\}$ and $\{f_4, f_5\}$. From the symmetry, we may assume that f_5 is adjacent to h_4. It follows from (3.2) applied to h_4 and h_5-f_3-f_4-f_5-f_1-h_5 that h_4 is non-adjacent to f_4. It follows from (3.2) applied to f_5 and H that f_5 is non-adjacent to h_3. By applying (3.4) to h_4, h_3 and F, h_3 has a neighbor in $V(F)$. Therefore, h_3 is adjacent to f_4. But now h_3 and h_4 are adjacent leaves for F that have different types, contradicting (3.3). This proves (3.9). □

3.3   Basic graphs

In the previous section, we showed that composite graphs in $\mathrm{Forb}(P_4^c, P_5, C_6)$, i.e. graphs that have a 5-gon with both a center and an anticenter, admit a quasi-homogeneous set decomposition. In this section, we will analyze basic graphs. It turns out that if a graph does not contain a 5-gon with both a center and an anticenter, then a 'dual' statement is also true: there is a no vertex that simultaneously serves as a center for some 5-gon in G and as an anticenter for some other 5-gon in G (we will prove this shortly). In particular, this implies that for every $v \in V(G)$, either $G|N(v)$ or $G|M(v)$ is perfect (and, equivalently, 1-narrow).

(3.12) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and suppose that no 5-gon has both a center and an anticenter. Then there do not exist v, A and B such that $v \in V(G)$, A and B are 5-gons in B, and v is a center for A and an anticenter for B.*

Proof. Suppose that v is a center for a 5-gon A and an anticenter for a 5-gon B. Since v is complete to $V(A)$ and anticomplete to $V(B)$, it follows that $V(A) \cap V(B) = \emptyset$. Let $a_1, a_2, \ldots, a_5$ and $b_1, b_2, \ldots, b_5$ be the vertices of A and B, respectively, in order.

(i)   *Every $x \in V(B)$ is a small attachment of A and all $x \in V(B)$ are of the same type.*

It follows from (3.2) that x is either an anticenter, or a small attachment, or a clone, or a center for A. Since G is basic, A does not have an anticenter and, hence, x is not an anticenter for A. Now suppose that x is a clone for A. It follows from (3.2) that v is adjacent to x, contradicting the fact that v is anticomplete to $V(B)$. This proves that every vertex in $V(B)$ is either a small attachment or a center for A.

Suppose that some vertex in $V(B)$ is complete to $V(A)$. Since B has no center, not all vertices in $V(B)$ are centers for A. Therefore, there are adjacent $y, z \in V(B)$ such that y is complete to $V(A)$ and z is not. Therefore, z is a small attachment of A. Let $a \in V(A)$ be a neighbor

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of z and let $a' \in V(A)$ be a non-neighbor of a. Since z is a small attachment of A, it follows that a' is non-adjacent to z. But now a-a'-z-v-y is an induced four-edge path, a contradiction. This proves that every vertex in $V(B)$ is a small attachment of A. Now suppose that not all vertices of $V(B)$ are of the same type with respect to A. Then there exist adjacent $b, b' \in V(B)$ such that b and b' are small attachments for A, but of different types, contradicting (3.3). This proves **(i)**.

(ii) *Let $x \in V(A)$. Then x is either a clone or an anticenter for B.*

Suppose that x is not a clone or an anticenter for B. Since G is basic, B does not have a center and, hence, x is not complete to $V(B)$. Then it follows from (3.2) that x is a small attachment of B. But now v is a neighbor of a small attachment of B and v has no neighbor in $V(B)$, contrary to (3.4). This proves **(ii)**.

From **(i)** and the symmetry, we may assume that all $b \in V(B)$ are of type 1. That is, for every $b \in V(B)$, b is either adjacent to a_1 and anticomplete to $\{a_2, a_3, a_4, a_5\}$, or b is adjacent to a_3 and a_4 and anticomplete to $\{a_1, a_2, a_5\}$. Since B does not have a center, at least one of the vertices of B is a leaf and at least one of them is a hat. From the symmetry, we may assume that b_1 is a leaf for A that is adjacent to a_1. Since from **(ii)** every vertex of A is either a clone or an anticenter for B, it follows that we may assume that a_1 is adjacent to b_4 and a_1 is anticomplete to $\{b_2, b_3\}$. Since a_1 is anticomplete to $\{b_2, b_3\}$, it follows from **(i)** that b_2 and b_3 are complete to $\{a_3, a_4\}$. Because b_1 and b_4 are leaves, it follows that $\{b_1, b_4\}$ is anticomplete to $\{a_3, a_4\}$. Therefore, it follows from (3.2) applied to a_3 and B that a_3 is a hat for B, contradicting **(ii)**. This proves (3.12). $\qquad\square$

We can now prove that

Theorem 3.13. *Every graph $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ is 2-narrow.*

Proof. We prove this by induction on $|V(G)|$. If G is perfect, then G is 1-narrow and there is nothing to prove. So we may assume that G is not perfect. From the fact that G has no induced four-edge antipath and no induced five-edge path, it follows that G contains a 5-gon. First suppose that G contains a 5-gon with a center and an anticenter. Then, by (3.9), G admits a quasi-homogeneous set decomposition (Z, A, C). Let G' be the graph obtained from $G|(A \cup C)$ by adding a vertex z anticomplete to A and complete C. By the induction hypothesis, G' and $G|Z$ are 2-narrow. It follows from Lemma 2.3 that G is 2-narrow. So we may assume that G has no 5-gon that has both a center and an anticenter. Let $v \in V(G)$. It follows from the induction hypothesis that $G|N(v)$ and $G|M(v)$ are both 2-narrow. Moreover, it follows from (3.12) that either $G|N(v)$ or $G|M(v)$ is perfect and hence 1-narrow. Since this is true for every $v \in V(G)$, it follows from Lemma 2.1 that G is 2-narrow. This proves Theorem 3.13. $\qquad\square$

4   Graphs in $\mathrm{Forb}(\mathbf{P}_4^c, \mathbf{P}_5)$

In this section, we will prove that every graph in $\mathrm{Forb}(P_4^c, P_5)$ is 3-narrow. Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and suppose that G does not contain a 6-gon with a center. Then it follows that $G|N(v) \in \mathrm{Forb}(P_4^c, P_5, C_6)$ for every $v \in V(G)$. In the previous section, we proved that therefore $G|N(v)$ is 2-narrow for every $v \in V(G)$. Now we may apply Lemma 2.1 to conclude that G is 3-narrow (for details, see the proof of Theorem 1.2 at the end of this section). The remaining case is when G does contain a 6-gon with a center. We deal with this case in (4.2). We will start with a lemma that deals with attachments of 6-gons.

(4.1) *Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and let H be a 6-gon in G with vertices $h_1, h_2, \ldots, h_6$ in order. Let $v \in V(G) \setminus V(H)$ and suppose that v has a neighbor and a non-neighbor in $V(H)$. Then, up to symmetry, either*

(x) *v is complete to $\{h_1, h_3, h_5\}$ and v is anticomplete to $\{h_2, h_4, h_6\}$, or*

(y) *v is complete to $\{h_3, h_6\}$, v is anticomplete to $\{h_1, h_2\}$ and v is either complete or anticomplete to $\{h_4, h_5\}$, or*

(z) *v is complete to $\{h_1, h_3\}$, anticomplete to $\{h_4, h_5, h_6\}$, and the adjacency between v and h_2 is arbitrary.*

Proof. We may assume that v is adjacent to h_1 and non-adjacent to h_2. Suppose that v is adjacent to h_3. Since h_1-h_2-h_3-h_4 is an induced path, and v is complete to $\{h_1, h_3\}$ and non-adjacent to h_2, it follows from (3.1) that v is non-adjacent to h_4. From the symmetry, it follows that v is non-adjacent to h_6. If v is adjacent to h_5, then (x) holds. If v is non-adjacent to h_5, then (z) holds. So we may assume that v is non-adjacent to h_3. If v is non-adjacent to h_4, then, since v-h_1-h_2-h_3-h_4-h_5 is not an induced five-edge path, it follows that v is adjacent to h_5 and (z) holds. So we may assume that v is adjacent to h_4. Because h_4-h_5-h_6-h_1 is an induced path and v is adjacent to h_1 and h_4, it follows from (3.1) that v is either complete or anticomplete to $\{h_5, h_6\}$. Therefore, (y) holds. This proves (4.1). $\qquad\square$

Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and let H be a 6-gon in G. We call a vertex $v \in V(G) \setminus V(H)$ an (x)-*vertex,* (y)-*vertex, or* (z)-*vertex for H* if v satisfies (x), (y), or (z) of (4.1), respectively. Let $z \in V(G) \setminus V(H)$ be a (z)-vertex for H. Then, there exists a unique vertex $h \in V(H)$ such that $H' = G|((V(H) \setminus \{h\}) \cup \{z\})$ is a 6-gon. We say that H' is *the 6-gon obtained from rerouting H through z.*

(4.2) *Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and suppose that G contains a 6-gon with a center. Then G admits a quasi-homogeneous set decomposition.*

Proof. Let H be a 6-gon with a center and let $h_1, h_2, \ldots, h_6$ be the vertices of H in order. Let C be the set of vertices that are complete to $V(H)$. Notice that $C \neq \emptyset$. Let $X, Y,$ and Z be the sets of (x)-vertices, (y)-vertices, and (z)-vertices for H, respectively.

(i)   *C is complete to $X \cup Y \cup Z$.*

Let $c \in C$ and $z \in Z$. Let H' be the 6-gon obtained from rerouting H through z. Then c has at least five neighbors in $V(H')$ and, hence, (4.1) implies that c is adjacent to z. This proves that C is complete to Z. Now let $x \in X$. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$ and anticomplete to $\{h_2, h_4, h_6\}$. Since h_6-h_1-x-h_3 is an induced path and c is complete to $\{h_1, h_3, h_6\}$, it follows from (3.1) that c is adjacent to x. Hence, C is complete to X. Next, let $y \in Y$. We may assume that y is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Then h_1-h_6-y-h_3 is an induced path and c is complete to $\{h_1, h_3, h_6\}$. It follows from (3.1) that y is adjacent to c and hence that Y is complete to C. This proves **(i)**.

Let Y' be the set of vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z)$ with a neighbor in Y. Let X' be the set of vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y')$ with a neighbor in X. Let X'' be the set of the vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y' \cup X')$ with a neighbor in X'. Let $A = V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y' \cup X' \cup X'')$. Since $(A \cup X' \cup X'' \cup Y') \cap (X \cup Y \cup Z \cup C) = \emptyset$, (4.1) implies that $A \cup Y' \cup X' \cup X''$ is anticomplete to $V(H)$. It follows from the definition of Y', X', X'', and A that $X' \cup X'' \cup A$ is anticomplete to Y, X is anticomplete to $X'' \cup A$, and X' is anticomplete to A.

(ii)   *Z is anticomplete to $A \cup X' \cup X'' \cup Y'$, Y' is anticomplete to $A \cup X' \cup X''$, A is anticomplete to X'', and X is anticomplete to Y.*

First, suppose that $z \in Z$ is adjacent to $a \in A \cup X' \cup X'' \cup Y'$. Let H' be obtained from rerouting H through z. Then it follows that a has exactly one neighbor in $V(H')$, contrary to (4.1). This proves that Z is anticomplete to $A \cup X' \cup X'' \cup Y'$.

Next, suppose that $y' \in Y'$ is adjacent to $a \in A \cup X' \cup X''$. Let $y \in Y$ be a neighbor of y'. We may assume that y is adjacent to h_3 and not to h_1 and h_2. Now h_1-h_2-h_3-y-y'-a is an induced five-edge path, a contradiction. This proves that Y' is anticomplete to $A \cup X' \cup X''$.

Next, suppose that $x'' \in X''$ is adjacent to $a \in A$. Then let $x' \in X'$ be a neighbor of x'' and let $x \in X$ be a neighbor of x'. From the symmetry, we may assume that x is adjacent to h_1 and not to h_2. Then h_2-h_1-x-x'-x''-a is an induced five-edge path, a contradiction. This proves that A is anticomplete to X''.

Finally, suppose that $x \in X$ and $y \in Y$ are adjacent. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$ and anticomplete to $\{h_2, h_4, h_6\}$, and that y is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Now, h_1-h_2-h_3-y is an induced path, x is complete to $\{h_1, h_3, y\}$ and x is non-adjacent to h_2, contrary to (3.1). This proves **(ii)**.

The following two claims deal with the case when $Y \neq \emptyset$.

(iii)   *Suppose that $Y \neq \emptyset$. Then there do not exist x, p, q such that $x \in X \cup Y$, $p, q \in X' \cup X'' \cup Y'$, and x-p-q is an induced path.*

Suppose that $Y \neq \emptyset$ and suppose that such x, p, q exist. First suppose that $x \in Y$. We may assume that x is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Now h_1-h_2-h_3-x-p-q is an induced five-edge path, a contradiction. We may therefore assume that $x \in X$. Let $y \in Y$. It follows from **(ii)** that y is non-adjacent to x. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$, y is complete to $\{h_3, h_6\}$ and y is anticomplete to $\{h_1, h_2\}$. Since q-p-x-h_1-h_6-y is not an induced five-edge path, it follows that y is adjacent to at least one of p and q. Because we already proved that no vertex in Y forms a two-edge induced path with p and q, it follows that y is complete to $\{p, q\}$. But now x-h_3-y-q is an induced path, p is complete to $\{x, y, q\}$, and p is non-adjacent to h_3, contrary to (3.1). This proves **(iii)**.

(iv) *If $Y \neq \emptyset$, then the lemma holds.*

Suppose that $Y \neq \emptyset$. We claim that $X'' = \emptyset$. For suppose that $x'' \in X''$. Then let $x' \in X'$ be a neighbor of x'', and let $x \in X$ be a neighbor of x'. Then x-x'-x'' is an induced path with $x \in X$ and $x', x'' \in X' \cup X''$, contrary to **(iii)**. This proves that $X'' = \emptyset$.

Let A' be the union of all the components K of $G|(X' \cup Y')$ such that C is not complete to K. Let $N = A \cup A'$ and $U = (V(H) \cup X \cup Y \cup Z \cup X' \cup Y') \setminus A'$. We claim that (U, N, C) is a quasi-homogeneous set decomposition of G. It follows from **(i)** and the definition of A' that C is complete to U. Let P be as in the definition of a quasi-homogeneous set decomposition and suppose that P is not perfect. Since P is an induced subgraph of G, it does not have an induced four-edge antipath or an induced five-edge path. It follows that P contains an induced cycle F of length five. Let $f_1, f_2, \ldots, f_5$ be the vertices of F in order. Let P^* be obtained from P by deleting all edges between U and N. It follows from Lemma 2.2 that P^* is perfect. Therefore, F is not an induced subgraph of P^*. It follows that there exist two adjacent vertices $a \in U$ and $b \in N$ such that $a, b \in V(F)$, say $f_1 = a$ and $f_2 = b$.

It follows from **(ii)** that A is anticomplete to U. Hence, because f_1 and f_2 are adjacent, it follows that $f_2 \notin A$ and therefore $f_2 \in A'$. It follows from the definition of A' that $f_1 \notin V(H) \cup X' \cup Y' \cup Z$ and hence $f_1 \in X \cup Y$. Now let us consider f_3. Since f_3 is adjacent to f_2, it follows that $f_3 \in X \cup Y \cup A' \cup C$. If $f_3 \in A'$, then f_1-f_2-f_3 is an induced path with $f_1 \in X \cup Y$ and $f_2, f_3 \in X' \cup Y'$, contrary to **(iii)**. Since $f_1 \in X \cup Y$, C is complete to $X \cup Y$, and f_3 is non-adjacent to f_1, it follows that $f_3 \notin C$, and therefore $f_3 \in X \cup Y$. Now let us consider f_4 and f_5. If both f_4 and f_5 are in $X' \cup Y'$, then f_3-f_4-f_5 is an induced path with $f_3 \in X \cup Y$ and $f_4, f_5 \in X' \cup Y'$, contrary to **(iii)**. Therefore, from the symmetry, we may assume that $f_4 \notin X' \cup Y'$. Since f_4 is adjacent to f_3, this implies that $f_4 \in V(H) \cup C \cup X \cup Y \cup Z$. Since f_4 is not adjacent to f_1 and C is complete to f_1, it follows that $f_4 \notin C$. Therefore, **(i)** implies that f_4 is complete to C. This proves that C is complete to $\{f_1, f_3, f_4\}$.

Let K' be the component of A' that contains f_2. We claim that f_1 is complete to K'. For suppose not. Because f_1 is adjacent to $f_2 \in K'$, it follows that there exist adjacent $k_1, k_2 \in K'$ such that f_1 is adjacent to k_1 and not to k_2. But now f_1-k_1-k_2 is an induced path with $f_1 \in X \cup Y$ and $k_1, k_2 \in X' \cup Y'$, contrary to **(iii)**. This proves that f_1 is complete to K' and, from the symmetry, that f_3 is complete to K'. Similarly, and since $V(H)$ is anticomplete to K', it follows

that f_4 is anticomplete to K'.

Since K' is not complete to C by the definition of A', we may choose $f'_2 \in K'$ and $c \in C$ such that f'_2 is non-adjacent to c (perhaps by choosing $f'_2 = f_2$). It follows from the above that f'_2 is adjacent to f_1 and f_3. Therefore, f_1-f'_2-f_3-f_4 is an induced path. It follows from the above that c is complete to $\{f_1, f_3, f_4\}$ and non-adjacent to f_2, contrary to (3.1). This proves **(iv)**.

In view of **(iv)**, we may from now on assume that no 6-gon in G has a (y)-vertex.

(v) *If $Z \neq \emptyset$, then the lemma holds.*

Suppose that $Z \neq \emptyset$. From the symmetry, we may assume that there exists $z \in Z$ such that z is adjacent to h_2 and h_6. Let Z'_1 be the set of vertices in Z that are adjacent to h_2 and h_6 and let $Z_1 = Z'_1 \cup \{h_1\}$. It follows from the definition of Z_1 that $|Z_1| \geq 2$. Let R be the set of vertices in $V(G) \setminus Z_1$ with a neighbor in Z_1 and let $S = V(G) \setminus (Z_1 \cup R)$. We claim that (Z_1, S, R) is a homogeneous set decomposition of G. For suppose not. Then there exist $v \in V(G) \setminus Z_1$ and $x, y \in Z_1$ such that v is adjacent to x and non-adjacent to y. It follows from the definition of Z_1 that $v \notin V(H)$. Let $H' = x$-h_2-h_3-$\ldots$-h_6-x. Since H' has no (y)-vertex and C is complete to Z_1 by **(i)**, it follows from (4.1) that v is either an (x)-vertex or a (z)-vertex for H'. It follows that v is anticomplete to h_4 and, since $v \notin Z_1$, v is adjacent to at least one of h_3, h_5. From the symmetry, we may assume that v is adjacent to h_3. It follows from the fact that v is either an (x)-vertex of a (z)-vertex for H', that v is non-adjacent to h_6. Since y-h_6-x-v-h_3-h_4 is not an induced five-edge path, it follows that x is adjacent to y. If v is non-adjacent to h_2, then x-h_3-y-v-h_2 is an induced four-edge antipath, a contradiction. Therefore, v is adjacent to h_2 and hence v is a (z)-vertex for H', and v is non-adjacent to h_5. But now the adjacency of v with respect to the 6-gon y-h_2-h_3-$\ldots$-h_6-y contradicts (4.1). This proves that (Z_1, R, S) is a homogeneous set decomposition of G. Since a homogeneous set decomposition is a special case of a quasi-homogeneous set decomposition, this proves **(v)**.

In view of **(v)**, we may from now on assume that $Z = \emptyset$. Let X_1 and X_2 be the vertices in X that are complete to $\{h_1, h_3, h_5\}$ and $\{h_2, h_4, h_6\}$, respectively.

(vi) *Suppose that $u \in X_1$ and $v \in X_2$ are non-adjacent. If $z \in V(G)$ is a common neighbor of u and v, then $z \in C$.*

By the definition of X_1 and X_2, u and v have no common neighbor in $V(H)$ and, thus, $z \in V(G) \setminus V(H)$. Consider the 6-gon $H' = u$-h_1-h_6-v-h_4-h_3-u. Since z is adjacent to u and v, it follows from (4.1) and the fact that no 6-gon in G has a (y)-vertex, that z is a center for H'. But now z has at least four neighbors in $V(H)$ and therefore, by (4.1) and since $Y = \emptyset$, $z \in C$. This proves **(vi)**.

Let $N = A \cup X' \cup X''$. We claim that $(V(H), N, C \cup X)$ is a quasi-homogeneous set decomposition of G. We observe that N is anticomplete to $V(H)$. Let P be as in the definition of a quasi-homogeneous set decomposition and suppose that P is not perfect. Since P is an induced subgraph

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of G, it does not have an induced four-edge antipath or an induced five-edge path. It follows that P contains an induced cycle F of length five. Let $f_1, f_2, \ldots, f_5$ be the vertices of F in order. Let P^* be obtained from P by adding all edges between $V(H)$ and X. It follows from Lemma 2.2 that P^* is perfect. Therefore, F is not an induced subgraph of P^*. It follows that there exist two non-adjacent vertices $a \in V(H)$ and $b \in X$ such that $a, b \in V(F)$, say $a = f_1$ and $f_3 = b$. From the symmetry, we may assume that $a = f_1 = h_1$. Since a and b are non-adjacent, it follows that $f_3 \in X_2$. Now let us consider f_5. From the fact that f_5 is adjacent to f_1, it follows that $f_5 \in V(H) \cup C \cup X_1$. Because f_3 is non-adjacent to f_5, it follows that $f_5 \notin V(H)$. Moreover, since C is complete to X_2, $f_3 \in X_2$ and f_3 is non-adjacent to f_5, it follows that $f_5 \notin C$ and, hence, $f_5 \in X_1$. Now f_4 is a common neighbor of $f_3 \in X_2$ and $f_5 \in X_1$. Therefore, by **(vi)**, it follows that $f_4 \in C$ and so f_4 is adjacent to f_1, a contradiction. This proves (4.2). $\qquad\square$

We are now in a position to prove Theorem 1.2:

Proof of Theorem 1.2. We prove the theorem by induction on $|V(G)|$. Let $G \in \mathrm{Forb}(P_4^c, P_5)$. Suppose first that G contains a 6-gon with a center. Then it follows from (4.2) that G admits a quasi-homogeneous set decomposition. Hence, it follows from Lemma 2.3 and the induction hypothesis that G is 3-narrow. So we may assume that G contains no 6-gon with a center. Now let $v \in V(G)$. Clearly, $G|N(v)$ does not have C_6 as an induced subgraph. Therefore, $G|N(v) \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and hence, by Theorem 3.13, $G|N(v)$ is 2-narrow. By the induction hypothesis, it follows that $G|M(v)$ is 3-narrow. Since this is true for every $v \in V(G)$, it follows from Lemma 2.1 that G is 3-narrow. This proves Theorem 1.2. $\qquad\square$

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