K_4-free graphs with no odd hole: even pairs and the circular chromatic number

Yori Zwols

Department of Industrial Engineering and Operations Research
Columbia University,
New York, NY 10027, USA

yz2198@columbia.edu

November 24, 2009

Abstract

An *odd hole* in a graph is an induced cycle of odd length at least five. In this paper we show that every imperfect K_4-free graph with no odd hole either is one of two basic graphs, or has an even pair or a clique cutset. We use this result to show that every K_4-free graph with no odd hole has circular chromatic number strictly smaller than 4. We also exhibit a sequence $\{H_n\}$ of such graphs with $\lim_{n\to\infty} \chi_c(H_n) = 4$.

1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. Let $G = (V, E)$ be a graph. We will let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. For $v \in V(G)$, let $N(v)$ denote the set of vertices that are adjacent to v. A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise non-adjacent vertices. For a set of vertices $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced by X and we let $G \backslash X = G|(V(G) \backslash X)$. An *induced path* in G is an induced subgraph of G that is a path. We will call an induced path *even* if it has an even number of edges, and *odd* if it has an odd number of edges. An *even pair* is a pair $\{u, v\}$ of vertices of G such that all induced paths in G from u to v are even. An *induced cycle* is an induced subgraph of G that is a cycle. An *odd hole* in a graph is an induced cycle of odd length at least five. A *cutset* X is a set of vertices such that $G \backslash X$ is disconnected. A *clique cutset* is a cutset that is a clique. For disjoint sets $A, B \subseteq V(G)$, we say that A is *complete to* B if every vertex in A is adjacent to every vertex in B, and that A is *anticomplete to* B if every vertex in A is non-adjacent to every vertex in B. A vertex v is said to be *complete* (*anticomplete*) to a set $A \subseteq V(G)$ if $\{v\}$ is complete (anticomplete) to A. We say that two disjoint sets A, B of vertices are *linked* if every vertex of A has a neighbor in B and every vertex of B has a neighbor in A. We denote by $\chi(G)$ the chromatic number of G, and by $\omega(G)$ the size of the largest clique in G. A graph G is called *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. A graph is called *imperfect* if it is not perfect.

For an integer $n \geq 1$, let K_n denote the complete graph on n vertices. For two coprime integers p, q with $p \geq 2q$, $K_{p/q}$ is a graph with vertex set $\{v_1, v_2, \dots, v_p\}$ such that v_i and v_j are adjacent if and only if $q \leq |i - j| \leq p - q$ for $i, j \in \{1, 2, \dots, p\}$. We call such a graph a *circular* p/q*-clique.* As special cases of circular cliques, let us define $\bar{C}_7 = K_{7/2}$ and $T_{11} = K_{11/3}$.

For a positive real number r and a graph G, a *circular* r*-coloring of* G is a function $c : V(G) \to [0, r)$ such that $1 \leq |c(u) - c(v)| \leq r - 1$ whenever $uv \in E(G)$. The *circular chromatic number of* G, denoted $\chi_c(G)$, is defined by $\chi_c(G) = \inf\{r : G \text{ has a circular } r\text{-coloring}\}$. The circular chromatic number was introduced by A. Vince in [7] as a refinement of the usual chromatic number of a graph. For clarity we note that in contrast to the usual (vertex) chromatic number, the circular chromatic is not necessarily an integer and in fact is in general a rational number [8]. It follows from the definition of $\chi_c(G)$ that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. We call a circular $\chi_c(G)$-coloring an *optimal circular coloring* of G. It was shown in [7] that an optimal circular coloring always exists. We refer to [8, 9] for good surveys on the circular chromatic number.

This paper deals with K_4-free graphs with no odd holes and consists of two parts. In the first part (Section 2), we will be interested in even pairs in K_4-free graphs with no odd holes. Our interest in even pairs originates in the fact that they are useful in graph coloring: contracting (see Section 3) an even pair in a graph does not change its chromatic number [4]. We will use the characterization of K_4-free graphs with no odd holes that was given in [3] to prove the following theorem about the structure of K_4-free graphs with no odd hole and no even pair:

Theorem 1.1. *Suppose that G is an imperfect K_4-free graph with no odd hole and no even pair. Then either G is isomorphic to one of $\{T_{11}, \bar{C}_7\}$, or G has a clique cutset.*

The second part of this paper is inspired by the following theorem from [3]:

Theorem. *Let G be a K_4-free graph with no odd hole. Then $\chi(G) \leq 4$.*

From this and the fact that $\chi_c(G) \leq \chi(G)$ for every graph G, it follows that the $\chi_c(G) \leq 4$ for every K_4-free graph G with no odd hole. In Section 3, we will use the results in Section 2 in conjuction with a linear programming argument to prove that this inequality is in fact strict:

Theorem 1.2. *Let G be a K_4-free graph with no odd hole. Then $\chi_c(G) < 4$.*

In Subsection 3.3 we will construct an infinite family of graphs whose circular chromatic number is arbitrarily close to 4, demonstrating that the bound given in Theorem 1.2 is tight.

2 Even pairs in K_4-free graphs with no odd hole

In order to prove Theorem 1.1, we will use the following structural theorem which is an immediate consequence of **3.1** and **9.1** in [3]. The definitions of the harmonious cutset, graphs of T_{11} type and graphs of the two heptagram types will be postponed until they are needed (they can also be found in [3]).

Theorem 2.1. [3] *Let G be an imperfect K_4-free graph with no odd hole. Then either G has a harmonious cutset, or G is of T_{11} type, or G is of the first or second heptagram type.*

We will analyze the presence of even pairs in K_4-free graphs with no odd hole by looking at the different outcomes of Theorem 2.1.

We say that a vertex u *dominates* another vertex v if $N(v) \subseteq N(u)$. We will repeatedly use the following observation:

Claim 2.2. *Let G be a graph and let $u, v \in V(G)$ be two non-adjacent vertices. If u dominates v, then $\{u, v\}$ is an even pair in G.*

Proof. We claim that every induced path from u to v in G has exactly two edges. For suppose that there exists a path Q from u to v in G of length other than two. Since u and v are non-adjacent, this implies that Q has at least three edges. Let u', v' be the neighbors of u, v, respectively on Q. Since Q is induced and has at least three edges, u' is non-adjacent to v and v' is non-adjacent to u. But this contradicts the fact that u dominates v. This proves Claim 2.2. □

Throughout this section we will use a slightly non-standard definition of a partition: for a set X, a *partition of* X is a collection $\{X_i\}_{i=1}^k$ of subsets of X such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^k X_i = X$. We stress that, in contrast to the usual definition of a partition, we do not require the sets of a partition to be non-empty (unless explicitly stated).

2.1 Harmonious cutsets and graphs of T_{11} type

We call a cutset X in G *harmonious* if X can be partitioned into disjoint non-empty stable sets $X_1, X_2, \ldots, X_k$ and

(S1) for all $i, j \in \{1, 2, \ldots, k\}$, if P is an induced path in G with one end in X_i and the other end in X_j, then P is even if $i = j$ and odd otherwise, and

(S2) if $k \geq 3$, then $X_1, X_2, \ldots, X_k$ are pairwise complete to each other.

We call a cutset X in G a *special odd cutset* if $|X| = 2$, the two vertices in X are non-adjacent and every induced path in G between them is odd. The following lemma almost proves Theorem 1.1 for graphs with harmonious cutsets. The only undesired outcome is the special odd cutset, which will be handled later, in Section 2.3.

Lemma 2.3. *Let G be a graph with a harmonious cutset. Then either G has an even pair, or G has a clique cutset, or G has a special odd cutset.*

Proof. Let $X, X_1, \ldots, X_k$ be as in the definition of the harmonious cutset. If $|X_i| \geq 2$ for some $i \in \{1, 2, \ldots, k\}$, any pair $\{u, v\} \subseteq X_i$ is an even pair by (S1). We may therefore assume that $|X_i| = 1$ for all $i \in \{1, 2, \ldots, k\}$. We may assume that X is not a clique, because if it is, then X is a clique cutset of size k and the lemma holds. It follows from (S1) and (S2) that X consists of two non-adjacent vertices such that every induced path between them is odd, so that X is a special odd cutset. This proves Lemma 2.3. □

We say that a graph G is *of T_{11} type* if $V(G)$ can be partitioned into stable sets $W_1, W_2, \ldots, W_{11}$ such that W_i is complete to W_j if and only if $3 \leq |i - j| \leq 8$ and anticomplete otherwise.

Lemma 2.4. *Suppose that G is a graph of T_{11} type. Then either G has an even pair, or G is isomorphic to T_{11}.*

Proof. Let $W_1, W_2, \ldots, W_{11}$ be as in the definition of a graph of T_{11} type. If $|W_i| = 1$ for all $i \in \{1, 2, \ldots, 11\}$, then G is isomorphic to T_{11}. Therefore, from the symmetry, we may assume that $|W_1| \geq 2$. Let $u, v \in W_1$ be distinct. Since $N(u) = N(v)$, it follows that u dominates v and hence from Claim 2.2 it follows that $\{u, v\}$ is an even pair. This proves Lemma 2.4. □

This handles the first two outcomes of Theorem 2.1. The next section is devoted to handling graphs of the first and second heptagram type.

2.2 Graphs of the first and second heptagram type

In this section, we will show that almost all graphs of the first and second heptagram type have an even pair, with graphs isomorphic to $\bar{C}_7$ as the only exception.

2.2.1 Graphs of the first heptagram type

We say that a graph G is of the *first heptagram type* if there exist $t \geq 1$ and a partition of $V(G)$ into ten stable sets $W_1, W_2, \ldots, W_7, Y_2, Y_4, Y_7$ where Y_4, Y_7 may be empty but the other sets are non-empty, such that, with index arithmetic modulo 7:

(A1) for $1 \leq i \leq 7$, W_i is complete to W_{i+2} and anticomplete to W_{i+3}

(A2) for $i \in \{3, 4, 6, 7\}$, W_i is complete to W_{i+1}, and for $i = 1, 2$, W_i, W_{i+1} are linked; and every vertex in W_2 is complete to one of W_1, W_3

(A3) for $i = 4, 7$, every vertex in Y_i is complete to $W_{i+3} \cup W_{i-3}$, has a neighbor in W_i, and has no neighbor in W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}

(A4) Y_2, Y_4, Y_7 are pairwise anticomplete

(A5) there is a non-empty subset $C \subseteq W_2$ such that C is complete to $W_1 \cup W_3$, and Y_2, C are linked, and Y_2 is anticomplete to $W_2 \setminus C$

(A6) there exist partitions $M_0, M_1, \ldots, M_t$ of W_5 and $N_0, N_1, \ldots, N_t$ of W_6 where M_0, N_0 may be empty but the other sets are non-empty, such that for $1 \leq i \leq t$, M_i is complete to N_i, M_i is anticomplete to $W_6 \setminus N_i$, $W_5 \setminus M_i$ is anticomplete to N_i, and M_0, N_0 are linked (and consequently W_5, W_6 are linked)

(A7) there is a partition $X_1, X_2, \ldots, X_t$ of Y_2 where $X_1, X_2, \ldots, X_t$ are all non-empty, such that for $1 \leq i \leq t$, X_i is complete to $M_i \cup N_i$, and anticomplete to each of $W_5 \setminus M_i, W_6 \setminus N_i, W_7, W_1, W_3, W_4$.

Graphs of the first heptagram type trivially have an even pair:

Lemma 2.5. *Let G be a graph of the first heptagram type. Then G has an even pair.*

Proof. Let $W_2, W_4, W_5, W_6, M_1, N_1, X_1$ be as in the definition of the first heptagram type. Let $u \in X_1$ and $v \in W_4$. It follows from (A1) and (A2) that v is complete to $W_2 \cup W_5 \cup W_6$. Moreover, it follows from (A3), (A4), (A6) and (A7) that $N(u) \subseteq M_1 \cup N_1 \cup W_2 \subseteq W_2 \cup W_5 \cup W_6 \subseteq N(v)$. Therefore, v dominates u and it follows from Claim 2.2 that $\{u, v\}$ is an even pair in G. This proves Lemma 2.5. $\square$

2.2.2 Graphs of the second heptagram type

Before we define the second heptagram type, let us say that a triple (W_1, W_2, W_3) of disjoint stable sets in G is a *crescent* if the following properties hold:

(C1) if $v_i \in W_i$ for $i = 1, 2, 3$ and v_2 is adjacent to both v_1 and v_3, then v_1 is adjacent to v_3

(C2) if $v_i \in W_i$ for $i = 1, 2, 3$ and v_2 is non-adjacent to both v_1 and v_3, then v_1 is non-adjacent to v_3.

We say that a graph G is of the *second heptagram type* if $V(G)$ can be partitioned into fourteen stable sets $W_1, W_2, \ldots, W_7$ and $Y_1, Y_2, \ldots, Y_7$ where $W_1, W_2, \ldots, W_7$ are non-empty but $Y_1, Y_2, \ldots, Y_7$ may be empty and the following properties (where arithmetic is modulo 7) hold:

(B1) for $1 \leq i \leq 7$, W_i is anticomplete to W_{i+3}

(B2) for $2 \leq i \leq 7$, W_i is complete to W_{i+2}, and W_1, W_2, W_3 are pairwise linked

(B3) (W_1, W_2, W_3) is a crescent, and if W_1 is not complete to W_3 then $Y_2, Y_5, Y_6 = \emptyset$

(B4) for $i \in \{3,4,6,7\}$, W_i is complete to W_{i+1}; W_5, W_6 are linked

(B5) for $1 \leq i \leq 7$, every vertex in Y_i is complete to $W_{i+3} \cup W_{i-3}$, has a neighbor in W_i, and has no neighbor in $W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}$

(B6) for $1 \leq i \leq 7$, every vertex in W_i with a neighbor in Y_i is complete to $W_{i+1} \cup W_{i-1}$

(B7) for $1 \leq i \leq 7$, Y_i is complete to Y_{i+1} and anticomplete to $Y_{i+2} \cup Y_{i+3}$

(B8) for $1 \leq i \leq 7$, at least one of Y_i, Y_{i+1}, Y_{i+2} is empty.

For distinct vertices $u, v, u', v' \in V(G)$, we say that the ordered pairs (u,v) and (u',v') are *friends* in G if u is adjacent to u', v is adjacent to v', u is non-adjacent to v' and v is non-adjacent to u'. For two disjoint sets $A, B \subseteq V(G)$, we say that A is *friendly with* B if there exists $u, v \in A$ and $u', v' \in B$ such that (u,v) and (u',v') are friends. Note that being friendly is a symmetric relationship so that we can speak of two sets A and B being *friendly with each other*.

We distinguish between graphs of the second heptagram type that have sets W_i and W_{i+1} that are friendly with each other and ones that do not have such sets. Note that from (B4) it follows that it suffices to look for such friendly sets for $i \in \{1,2,5\}$.

Graphs of the second heptagram type with friends

We will start with two claims about graphs of the second heptagram type that have a friendly pair in consecutive W_i's and then deduce that such graphs have an even pair.

Claim 2.6. *Let (W_1, W_2, W_3) be a crescent, suppose that W_1, W_2, W_3 are pairwise linked and that $(u,v) \in W_1 \times W_1$ and $(u',v') \in W_2 \times W_2$ are friends. Then W_3 can be partitioned into sets $S \neq \emptyset$ and T such that $\{u,v,u',v'\}$ is complete to S and anticomplete to T.*

Proof. Let $x \in W_3$. Suppose that x is adjacent to one of $\{u,v\}$. From the symmetry, we may assume that x is adjacent to u. By (C2) applied to u, v', x, it follows that x is adjacent to v'. By (C1) applied to v, v', x, it follows that x is adjacent to v. So x is complete to $\{u,v\}$. Therefore, W_3 can be partitioned into sets S and T such that S is complete to $\{u,v\}$ and T is anticomplete to $\{u,v\}$. It follows from the fact that W_1 and W_3 are linked that $S \neq \emptyset$. Now let $s \in S$. By (C2) applied to u, v', s and to v, u', s, it follows that s is adjacent to u', v' and hence S is complete to $\{u,v,u',v'\}$. Next, let $t \in T$. By (C1) applied to u, u', t and to v, v', t, it follows that t is non-adjacent to u', v' and hence T is anticomplete to $\{u,v,u',v'\}$. This proves Claim 2.6. □

Claim 2.7. *Let G be a graph of the second heptagram type. Let $i \in \{1,2,5\}$ and suppose that $(u,v) \in W_i \times W_i$ and $(u',v') \in W_{i+1} \times W_{i+1}$ are friends. Then $N(u) \backslash W_{i+1} = N(v) \backslash W_{i+1}$ and $N(u') \setminus W_i = N(v') \setminus W_i$.*

Proof. From the symmetry, we may assume that $i \in \{1,5\}$. First suppose that $i = 5$. It suffices to show that $N(u) \backslash W_6 = N(v) \backslash W_6$. We first note that from (B1) and (B5), it follows that $N(u) \backslash W_6$ and $N(v) \backslash W_6$ are subsets of $W_3 \cup W_4 \cup W_7 \cup Y_1 \cup Y_2 \cup Y_5$. It follows from (B2) and (B5) that u and v are complete to $W_3 \cup W_4 \cup W_7 \cup Y_1 \cup Y_2$. It follows from (B6) and the fact that (u,v) and (u',v') are friends that $\{u,v\}$ is anticomplete to Y_5. Therefore, $N(u)\backslash W_6 = N(v)\backslash W_6 = W_3 \cup W_4 \cup W_7 \cup Y_1 \cup Y_2$. This proves the claim when $i = 5$.

So we may assume that $i = 1$. We first claim that $N(u)\backslash W_2 = N(v)\backslash W_2$. It follows from (B1) and (B5) that $N(u)\backslash W_2$ and $N(v)\backslash W_2$ are subsets of $W_3 \cup W_6 \cup W_7 \cup Y_1 \cup Y_4 \cup Y_5$. It follows from (B2) and (B5) that u and v are complete to $W_6 \cup W_7 \cup Y_4 \cup Y_5$. It follows from (B6) and the fact that (u, v) and (u', v') are friends that $\{u, v\}$ is anticomplete to Y_1. Finally, it follows from Claim 2.6 that W_3 can be partitioned into sets S, T such that S is complete to $\{u, v\}$ and T is anticomplete to $\{u, v\}$. Therefore, $N(u)\backslash W_2 = N(v)\backslash W_2 = W_6 \cup W_7 \cup Y_4 \cup Y_5 \cup S$. Next, we claim that $N(u') \setminus W_1 = N(v')\backslash W_1$. It follows from (B1) and (B5) that $N(u')\backslash W_1$ and $N(v')\backslash W_1$ are subsets of $W_3 \cup W_4 \cup W_7 \cup Y_2 \cup Y_5 \cup Y_6$. It follows from (B2) and (B5) that u' and v' are complete to $W_4 \cup W_7 \cup Y_5 \cup Y_6$. It follows from (B6) and the fact that (u, v) and (u', v') are friends that $\{u, v\}$ is anticomplete to Y_2. Finally, it follows from Claim 2.6 that W_3 can be partitioned into sets S, T such that S is complete to $\{u', v'\}$ and T is anticomplete to $\{u', v'\}$. In particular, $N(u') \cap W_3 = N(v') \cap W_3$. Therefore, $N(u)\backslash W_2 = N(v)\backslash W_2 = W_4 \cup W_7 \cup Y_5 \cup Y_6 \cup S$, thereby completing the proof of Claim 2.7. □

This enables us to find even pairs:

Lemma 2.8. *Let G be a graph of the second heptagram type. Suppose that W_i is friendly with W_{i+1} for some $i \in \{1, 2, \ldots, 7\}$. Then G has an even pair.*

Proof. It follows from (B4) that $i \in \{1, 2, 5\}$. Let $(u, v) \in W_i \times W_i$ and $(u', v') \in W_{i+1} \times W_{i+1}$ be friends. For $q, q' \in \{1, 2, \ldots, 7\}$, let $\mathcal{P}_{q,q'}$ be the set of odd induced paths with endpoints $a, b \in W_q$ such that there exist $a', b' \in W_{q'}$ such that (a, b) and (a', b') are friends. Let $\mathcal{P} = \mathcal{P}_{i,i+1} \cup \mathcal{P}_{i+1,i}$. We will show that $\mathcal{P} = \emptyset$, implying that there exists no odd induced path from u to v, and therefore that $\{u, v\}$ is an even pair.

Suppose for a contradiction that $\mathcal{P} \neq \emptyset$. Then there exists $P \in \mathcal{P}$ such that $|V(P)|$ is minimum. Let a and b be the endpoints of P. Let $\{j, k\} = \{i, i + 1\}$ be such that $P \in \mathcal{P}_{j,k}$. It follows that $a, b \in W_j$. Let a' and b' be the neighbors of a and b, respectively, in P. It follows from the fact that P is an induced odd path that (a, b) and (a', b') are friends. From this and the fact that $N(a)\backslash W_k = N(b)\backslash W_k$ by Claim 2.7, it follows that $a', b' \in W_k$. Now construct the path P' from P by deleting the endpoints a and b. Clearly, $P' \in \mathcal{P}$, but $|V(P')| < |V(P)|$, a contradiction. This proves that $\mathcal{P} = \emptyset$, thereby completing the proof of Lemma 2.8. □

Graphs of the second heptagram type with no friends

We will now turn to graphs with no friends:

Claim 2.9. *Let G be a graph of the second heptagram type. Let $i \in \{1, 2, \ldots, 7\}$ and suppose that W_i is not friendly with W_{i+1}. Then there exists $u \in W_i$ that is complete to $W_{i+1} \cup W_{i+2}$.*

Proof. Let $u \in W_i$ be a vertex with a maximum number of neighbors in W_{i+1}. We first claim that u is complete to W_{i+1}. For suppose that there exists $v' \in W_{i+1}$ that is not adjacent to u. From (B2) and (B4), it follows that v' has a neighbor $v \in W_i$. By the choice of u, there exists $u' \in W_{i+1}$ that is adjacent to u but not to v. But now (u, v) is friends with (u', v'), contradicting the fact that W_i is not friendly with W_{i+1}. Therefore u is complete to W_{i+1}. If $i \neq 1$, it follows from (B2) that u is complete to W_{i+2}. If $i = 1$, let $x \in W_3$ be given. From (B2) it follows that x has a neighbor $v \in W_2$. From (C1) applied to u, v, x, it follows that u and x are adjacent. Therefore u is complete to W_{i+2}. This proves Claim 2.9. □

Lemma 2.10. *Let G be a graph of the second heptagram type. Suppose that for each $i \in \{1, 2, \ldots, 7\}$, W_i is not friendly with W_{i+1}. Then either G has an even pair, or G is isomorphic to $\bar{C}_7$.*

Proof. We may assume that G is not isomorphic to $\bar{C}_7$.

(i) *Let $i \in \{1, 2, \ldots, 7\}$. Then there exists $u \in W_i$ such that u is complete to $W_{i-2} \cup W_{i+1} \cup W_{i+2}$ and, if $i \neq 2$, then u is complete to $W_{i-2} \cup W_{i-1} \cup W_{i+1} \cup W_{i+2}$.*

From the symmetry, we may assume that $i \in \{1, 2, 4, 5\}$. It follows from Claim 2.9 that there exists $u \in W_i$ such that u is complete to $W_{i+1} \cup W_{i+2}$. It follows from (B2) that W_i is complete to W_{i-2} (since $i \neq 3$). This proves the claim if $i = 2$. So we may assume that $i \neq 2$. It follows from (B4) that u is complete to W_{i-1}. This proves **(i)**.

(ii) *If $Y_i \neq \emptyset$ for some $i \in \{1, 2, \ldots, 7\}$, then G has an even pair.*

Choose any $y \in Y_i$. It follows from property (B8) and the symmetry that we may assume that $Y_{i-1} = \emptyset$. By **(i)**, we may choose $u \in W_{i-2}$ such that u is complete to $W_i \cup W_{i-3} \cup W_{i+3}$. It follows from (B5) that u is complete to Y_{i+1}. Therefore, by (B5) and (B7), $N(y) \subseteq W_i \cup W_{i-3} \cup W_{i+3} \cup Y_{i+1} \subseteq N(u)$, and u and y are non-adjacent. Hence it follows from Claim 2.2 that $\{u, y\}$ is an even pair. This proves **(ii)**.

In the light of **(ii)**, we may now assume that $Y_i = \emptyset$ for all $i \in \{1, 2, \ldots, 7\}$. If $|W_i| \geq 2$ for some $i \in \{1, 3, 4, \ldots, 7\}$, then by **(i)** we may choose $u \in W_i$ such that u is complete to $W_{i-2} \cup W_{i-1} \cup W_{i+1} \cup W_{i+2}$ and choose any other $v \in W_i$. It follows that u dominates v and hence, by Claim 2.2, that $\{u, v\}$ is an even pair. So we may assume that $|W_i| = 1$ for every $i \in \{1, 3, 4, \ldots, 7\}$. Since G is not isomorphic to $\bar{C}_7$, it follows that $|W_2| \geq 2$. Now choose $u, v \in W_2$. It follows from the definition of the second heptagram type that $N(u) = N(v)$ and hence it follows from Claim 2.2 that $\{u, v\}$ is an even pair. This proves Lemma 2.10. □

2.3 Proof of Theorem 1.1

The lemmas from Sections 2.1 and 2.2 lead to the following structural result:

Lemma 2.11. *Suppose that G is an imperfect K_4-free graph with no odd hole. Then either*

1. *G is isomorphic to one of $\{T_{11}, \bar{C}_7\}$, or*

2. G has an even pair, or

3. G has a clique cutset, or

4. G has a special odd cutset.

Proof. It follows from Theorem 2.1 that either G has a harmonious cutset, or G is of T_{11} type, or G is of the first or the second heptagram type. If G has a harmonious cutset, is of T_{11} type, or is of the first heptagram type, then the result follows from Lemma 2.3, Lemma 2.4, Lemma 2.5, respectively. If G is of the second heptagram type and W_i friendly with W_{i+1} for some $i \in \{1, 2, \ldots, 7\}$, then the result follows from Lemma 2.8. If G is of the second heptagram type and no such i exists, then the result follows from Lemma 2.10. This proves Lemma 2.11. □

We finish the proof of Theorem 1.1 by showing that outcome 4 in Lemma 2.11 is redundant:

Proof of Theorem 1.1. Let G be an imperfect K_4-free graph with no odd hole with $|V(G)|$ minimum such that G has a special odd cutset $\{u, v\}$ and none of the outcomes 1, 2, 3 of Lemma 2.11 hold. Recall that, by the definition of a special odd cutset, u and v are non-adjacent in G and every path between u and v is odd. Let

$$\mathcal{V} = \left\{ (V_1, V_2) \;\middle|\; \begin{array}{c} V_1, V_2 \subsetneq V(G) \backslash \{u, v\}, \\ (V_1, V_2) \text{ is a partition of } V(G) \backslash \{u, v\}, \\ \text{and } V_1 \text{ is anticomplete to } V_2 \end{array} \right\}.$$

Since $\{u, v\}$ is a cutset, $\mathcal{V}$ is non-empty. For every partition $(V_1, V_2) \in \mathcal{V}$ and for $i \in \{1, 2\}$, let $G_i(V_1, V_2)$ be the graph constructed from $G|(V_i \cup \{u, v\})$ by adding an edge between u and v. Note that since $\{u, v\}$ is a special odd cutset, $G_1(V_1, V_2)$ and $G_2(V_1, V_2)$ are both K_4-free graphs with no odd hole. From $\mathcal{V}$ choose a partition (V_1, V_2) such that the graph $G_1^* = G_1(V_1, V_2)$ is imperfect and, subject to this, $|V_1|$ is minimum. Such a partition exists because it was shown in [6] that if G_1 and G_2 defined as above are both perfect, then G is also perfect, contrary to the assumption that G is imperfect. Since G_1^* is imperfect, it follows from Lemma 2.11 and the choice of G that either G_1^* is isomorphic to one of T_{11} and $\bar{C}_7$, or G_1^* has an even pair, or G_1^* has a clique cutset.

First suppose that G_1^* isomorphic to T_{11} or $\bar{C}_7$. Since in both T_{11} and $\bar{C}_7$ every two adjacent vertices have a common neighbor, u and v have a common neighbor $x \in V_1$. It follows that $\{u, x, v\}$ induces a two-edge path in G from u to v, contradicting the fact that every path from u to v is odd. So G_1^* is not isomorphic ot T_{11} or to $\bar{C}_7$.

Next suppose that G_1^* has an even pair $\{a, b\}$. Let P be an induced path from a to b in G. We claim that P is even. If $\{u, v\} \not\subseteq V(P)$, then, because $a, b \in V(G_1^*)$, it follows that P is also an induced path in G_1^*, and hence that P is even. So we may assume that $\{u, v\} \subseteq V(P)$. From the symmetry in u, v, we may assume that there are induced paths P_1, P_2, P_3 in G such that $P = a\text{-}P_1\text{-}u\text{-}P_2\text{-}v\text{-}P_3\text{-}b$. Since $\{a, b\}$ is an even pair in G_1^* and $G|((V(P) \setminus V(P_2) \cup \{u, v\})$ is an induced path between a and b in G_1^*, it follows that $|E(P_1)| + |E(P_3)|$ is odd. Moreover, since $\{u, v\}$ is a special odd cutset and P_2 is an induced path between u and v in G, it follows that $|E(P_2)|$ is odd. Hence, $|E(P)| = |E(P_1)| + |E(P_2)| + |E(P_3)|$ is even. This proves that every induced path in

G between a and b is even and, therefore, that $\{a, b\}$ is an even pair in G, contrary to our assumption that G does not have an even pair.

Finally assume that G_1^* has a clique cutset X. Let (C_1, C_2) with $C_1, C_2 \subsetneq V(G_1^*)\backslash X$ be a partition of $V(G_1^*)\backslash X$ such that C_1 is anticomplete to C_2 in G_1^*. If at most one of u and v is an element of X, then X is also a clique cutset in G, contrary to the assumption that G does not satisfy outcome 3 of Lemma 2.11. Therefore $\{u, v\} \subseteq X$. Since u and v do not have common neighbors (otherwise there exists a two-edge path in G between u and v) and X is a clique, it follows that $X = \{u, v\}$. Since G_1^* is imperfect, at least one of $G_1^*|(C_1 \cup \{u, v\})$ and $G_1^*|(C_2 \cup \{u, v\})$ is (as shown in [6]). By the symmetry we may assume that $G_1^*|(C_1 \cup \{u, v\})$ is imperfect. But now $(C_1, C_2 \cup V_2) \in \mathcal{V}$ and $|C_1| < |V_1|$, contradicting the minimality of V_1.

This proves Theorem 1.1. □

3 Circular coloring

In this section we will use the outcomes of Theorem 1.1 to show that every K_4-free graph with no odd hole has circular chromatic number strictly less than 4. It was shown in [7] that for coprime integers p, q with $p \geq 2q$, $\chi_c(K_{p/q}) = p/q$ and hence in particular we have that $\chi_c(\bar{C}_7) = 7/2$ and $\chi_c(T_{11}) = 11/3$. This immediately handles the first outcome of Theorem 1.1.

For a graph G and two vertices $x, y \in V(G)$, let G/xy be the graph obtained by deleting x and y and adding a new vertex xy adjacent to precisely $N(x) \cup N(y)$. This operation is called *contracting the pair* $\{x, y\}$. As said in the introduction, contracting even pairs does not decrease the circular chromatic number of the graph. In fact, contracting non-adjacent vertices does not decrease the circular chromatic number:

Lemma 3.1. *Let G be a graph and let $x, y \in V(G)$ be non-adjacent. Then, $\chi_c(G) \leq \chi_c(G/xy)$.*

Proof. Let $r = \chi_c(G/xy)$ and let $c : V(G/xy) \to [0, r)$ be a circular r-coloring of G/xy. It is straightforward to verify that $c' : V(G) \to [0, r)$ defined by $c'(u) = c(u)$ for all $u \in V(G/xy)\backslash\{xy\}$ and $c'(x) = c'(y) = c(xy)$ is a circular r-coloring of G. It follows that $\chi_c(G) \leq r$. This proves Lemma 3.1. □

(We note that the graph H_n defined in the last part of Section 3.3 shows that the inequality in this Lemma is really an inequality, as opposed to its usual chromatic number counterpart [5] in which equality holds. This follows from the fact that glueing two graphs on an edge can be viewed as twice contracting an even pair.)

The following two subsections will be devoted to handling clique cutsets. We will start with a result about optimal circular coloring of large circular cliques. This will allow us to prove a lemma on the circular chromatic number of graphs that are obtained by "glueing" two copies of $K_{(tk-1)/k}$ (where $t \geq 3$ and $k \geq 1$ are integers) on an appropriately chosen clique. (The glueing operation will be made precise.) The result is the basis for showing that glueing two graphs that have circular chromatic number strictly less than 4 does not increase the circular chromatic number beyond 4, as long as we glue on triangles and edges. This handles clique cutsets, the second outcome of Theorem 1.1.

Throughout this section, we will use the following equivalent definition of the circular chromatic number [1]. Let G_1 and G_2 be graphs. We say that a function $f : V(G_1) \to V(G_2)$ is a *homomorphism from G_1 to G_2* if $f(u)f(v) \in E(G_2)$ whenever $uv \in E(G_1)$. For finite graphs, $\chi_c(G) = \inf\{p/q : \text{there exists a homomorphism from } G \text{ to } K_{p/q}\}$. The following theorem was implicitly proved in [1]:

Theorem 3.2. [1] *For coprime integers p, q with $p \geq 2q$, a graph G is circular p/q-colorable if and only if there exists a homomorphism from G to $K_{p/q}$.*

3.1 χ-Critical circular cliques in optimal circular colorings

Let G be a graph. For coprime integers p, q with $p \geq 2q$, we call an induced subgraph H of G a *χ-critical circular clique* if H is isomorphic to $K_{p/q}$ and $\chi(G) - 1 < p/q < \chi(G)$. Note that, by definition, for any χ-critical circular clique H, $\chi_c(H)$ is a non-integer larger than 2. We will start with a lemma that states that if $\chi_c(G) < \chi(G)$ then every χ-critical circular clique in G is optimally circularly colored either "clockwise" or "counterclockwise". To make this precise, let us introduce some notation. For a real number $r > 2$ and real numbers a, b, let $[a, b]_r$ denote the closed interval from a to b in the cyclic group $\mathbb{R}/r\mathbb{Z}$. That is, writing $a' = a \pmod r$, $b' = b \pmod r$,

$$\text{if } a \leq b\text{: } [a, b]_r = \begin{cases} [a', b'] & \text{if } a' \leq b' \\ [b', r) \cup [0, a'] & \text{if } a' > b' \end{cases}$$

$$\text{if } a > b\text{: } [a, b]_r = [0, r) \backslash [b, a]_r.$$

Let the open interval $(a, b)_r$ and half-open intervals $[a, b)_r$, $(a, b]_r$ be defined in the obvious analogous way.

For a circular coloring $c : V(G) \to [0, r)$ of a graph G and $s \in \mathbb{R}$, define $T_s c : V(G) \to [0, r)$ by $T_s c(v) = c(v) + s \pmod r$. We say that an induced subgraph of G with vertices $\{v_1, v_2, \ldots, v_n\}$ is *circularly colored clockwise (with respect to the circular coloring c)* if there exists an $s \in \mathbb{R}$ such that $T_s c(v_1) \leq T_s c(v_2) \leq \cdots \leq T_s c(v_n)$. We call the function $r - c$ the *reversion* of the circular r-coloring c. We say that an induced subgraph is *circularly colored counterclockwise (w.r.t. c)* if it is circularly colored clockwise w.r.t. $r - c$. Note that if c is a circular r-coloring of G, then $T_s c$ and $r - c$ are circular r-colorings of G as well.

Lemma 3.3. *Let p, q be coprime integers with $p \geq 2q$. Let G be a graph with $\chi_c(G) < \chi(G)$, let H be a χ-critical circular clique in G with $\chi_c(H) = p/q$ and vertex set $\{v_1, v_2, \ldots, v_p\}$ such that $v_i v_j \in E(G)$ if and only if $q \leq |i - j| \leq p - q$. Let c be an optimal circular coloring of G. Then H is either circularly colored clockwise or circularly colored counterclockwise with respect to c.*

Proof. Let $r = \chi_c(G)$. Note that, by the existence of a χ-critical circular clique, $r > 2$. Observe that for every $j \in \{1, 2, \ldots, p\}$ it follows from the fact that v_j is adjacent to each of $\{v_{j+q}, v_{j+q+1}, \dots, v_{j+p-q}\}$ that

$$c(v_i) \in [c(v_j)+1, c(v_j)-1]_r \text{ for all } i \in \{v_{j+q}, \dots, v_{j+p-q}\}. \tag{1}$$

(i) *Let $j \in \{1, 2, \dots, p\}$ and let k be an integer such that $1 \le |k| \le q-1$. Then $c(v_{j+k}) \in (c(v_j)-1, c(v_j)+1)_r$.*

From the symmetry, it suffices to show this for $1 \le k \le q-1$. Let $s = \lfloor p/q \rfloor$ and note that $r-1 < \chi(G)-1 \le \lfloor p/q \rfloor = s$. Now suppose that $c(v_{j+k}) \in [c(v_j)+1, c(v_j)-1]_r$. Since $(s-1)q < p-q$, the set $K = \{v_{j+k}, v_{j+k+q}, \dots, v_{j+k+(s-1)q}\}$ is a clique of size s in G (see Figure 1). From (1) and the assumption that $c(v_{j+k}) \in [c(v_j)+1, c(v_j)-1]_r$, it follows that K satisfies $c(u) \in [c(v_j)+1, c(v_j)-1]_r$ for all $u \in K$. But note that the length of the interval $[c(v_j)+1, c(v_j)-1]_r$ is $r-2$. Therefore we can construct from c a circular $(r-1)$-coloring of $G|K$ by replacing the interval $[c(v_j)-1, c(v_j)+1]_r$ by an interval of length 1 and restricting c to K. But since $r-1 < s$, this contradicts the fact that $\chi_c(G|K) = s$. This proves **(i)**.

(ii) *Let $j \in \{1, 2, \dots, p\}$. Then up to reversion of c, for every $k \in \{1, 2, \dots, q-1\}$, $c(v_{j-k}) \in (c(v_j)-1, c(v_j)]_r$ and $c(v_{j+k}) \in [c(v_j), c(v_j)+1)_r$.*

By **(i)**, either $c(v_{j-q+1}) \in (c(v_j)-1, c(v_j)]_r$ or $c(v_{j-q+1}) \in [c(v_j), c(v_j)+1)_r$. By reversing c, we may assume that the former is the case. From **(i)** and the fact that v_{j-q+1} is adjacent to each of $v_{j+1}, \dots, v_{j+q-1}$, it follows that $c(v_{j+k}) \in [c(v_j), c(v_j)+1)_r$ for all $k \in \{1, 2, \dots, q-1\}$. In turn, since in particular $c(v_{j+q-1}) \in [c(v_j), c(v_j)+1)_r$, it follows by the same reasoning that $c(v_{j-k}) \in (c(v_j)-1, c(v_j)]_r$ for all $k \in \{1, 2, \dots, q-1\}$, proving **(ii)**.

We can now prove the lemma. Possibly by taking the reversion of c and considering $T_{-c(v_1)}c$ instead of c, we may assume that $c(v_1) = 0$ and that $0 \le c(v_2) < 1$. We prove by induction on n that $c(v_1) \le c(v_2) \le \cdots \le c(v_n)$. Note that this is true for $n = 2$. Suppose it is true for $n = N$, $2 \le N \le p-1$. We claim that $c(v_N) \le c(v_{N+1})$. Since by **(ii)** $c(v_{N-1}) \in (c(v_N)-1, c(v_N)]_r$, it also follows from **(ii)** that $c(v_{N+1}) \in [c(v_N), c(v_N)+1)_r$. If $c(v_N) < r-1$, then we are done. So we may assume

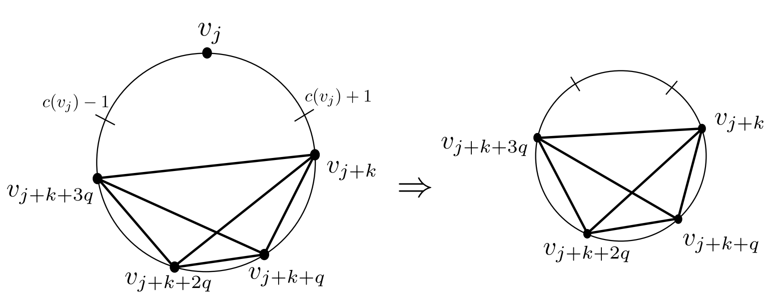

Figure 1: Part **(i)** of Lemma 3.3. The diagram on the left shows the "colors" assigned to $v_j, v_{j+k}, v_{j+k+q}, v_{j+k+2q}, v_{j+k+3q}$ on the circular interval $[0, r)_r$. The diagram on the right shows a circular $(r-1)$-coloring of $G|K$.

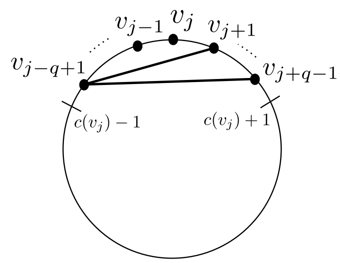

Figure 2: Part **(ii)** of Lemma 3.3.

that $c(v_N) \geq r-1 > 1$. It remains to show that $c(v_{N+1}) \notin [0,1]_r$. For suppose that $c(v_{N+1}) \in [0,1]_r$. It follows from the fact that $c(v_1) = 0$ and (1) that $N+1 < q+1$ or $N+1 > 1+p-q$. If $N+1 < 1+q$, then $N < q$ and hence, by **(ii)** and the induction hypothesis, it follows that $0 \leq c(v_N) < 1$, a contradiction. If $N+1 > 1+p-q$, then since $c(v_2) \in [0,1)$ and by **(ii)**, it follows that $c(v_{N+1}) \in (c(v_1)-1, c(v_1)]_r = (r-1, r]_r$, a contradiction. This completes the proof of Lemma 3.3. ☐

3.2 Circular coloring and clique cutsets

Let H_1 and H_2 be two graphs with disjoint vertex sets, let T_1 and T_2 be cliques in H_1 and H_2, respectively, with $|T_1| = |T_2|$, and let $f : T_1 \to T_2$ be a bijective mapping. We define the *clique sum of* (H_1, T_1) *and* (H_2, T_2) *through* f as the graph G with vertex set $V(G) = (V(H_1) \cup V(H_2)) \backslash T_2$ and in which $u, v \in V(G)$ are adjacent if

1. $\{u, v\} \subseteq V(H_1)$ and $uv \in E(H_1)$; or
2. $\{u, v\} \subseteq V(H_2 \backslash T_2)$ and $uv \in E(H_2)$; or
3. $u \in T_1, v \in V(H_2 \backslash T_2)$ and $f(u)v \in E(H_2)$;

and non-adjacent otherwise. Note that our definition of the clique sum is non-standard because we do not allow for deletion of clique edges. (However, this restriction is irrelevant for the analysis in this paper since deleting edges from a graph does not increase its circular chromatic number.)

For motivation of this section, we note that, by definition, $\chi_c(G)$ is the optimal value of the following optimization problem with decision variables $r, \{x_v\}_{v \in V(G)}$:

$$\begin{aligned} \chi_c(G) = \min\ & r \\ \text{s.t. }\ & 1 \leq |x_u - x_v| \leq r-1, && \text{for all } uv \in E(G) \\ & 0 \leq x_v \leq r, && \text{for all } v \in V(G). \end{aligned} \tag{2}$$

There is an obvious one-to-one correspondence between the feasible points of this problem and the circular colorings of G. However, this problem is hard to deal with in general because of the $|x_u - x_v|$ term. Nevertheless, if the ordering of $\{x_v\}$ is known for some optimal solution, then each term $|x_u - x_v|$ can be replaced by $x_u - x_v$ or $x_v - x_u$, depending on whether $x_u \geq x_v$ or $x_u \leq x_v$ in this optimal solution. Doing this turns the problem into a linear program, which is much easier to handle.

Suppose that G is a clique sum of two copies H_1 and H_2 of $K_{(tk-1)/k}$. Then from Lemma 3.3, it follows that if $\chi_c(G) < \chi(G)$, then for both H_1 and H_2 there are only two possible optimal circular colorings: clockwise and counterclockwise. From the symmetry, we may assume that H_1 is circularly colored clockwise and moreover that a (fixed) common vertex receives color 0. This means that we can recover the circular chromatic number by taking the minimum of the optimal values of two linear programs (corresponding to the cases where H_2 is circularly colored clockwise and circularly colored counterclockwise, respectively), as long as at least one of them has optimal value strictly less than $\chi(G)$. We use this idea to prove the following lemma. (In the proof of this lemma, we will in fact pick the correct one of the two linear programs.)

Lemma 3.4. *Let $t \geq 2$ and $k \geq 1$ be integers and let H_1, H_2 be two copies of $K_{(tk-1)/k}$ with disjoint vertex sets $\{u_1, u_2, \ldots, u_{tk-1}\}$ and $\{v_1, v_2, \ldots, v_{tk-1}\}$, respectively, such that $u_i u_j \in E(H_1) \iff v_i v_j \in E(H_2) \iff k \leq |i-j| \leq (t-1)k-1$. Let s, $a_1, a_2, \ldots, a_s$, $b_1, b_2, \ldots, b_s$ be integers such that $T_1 := \{u_{a_1}, u_{a_2}, \ldots, u_{a_s}\}$ and $T_2 := \{v_{b_1}, v_{b_2}, \ldots, v_{b_s}\}$ are cliques in H_1 and H_2, respectively, and assume that for every $j \in \{1, 2, \ldots, s\}$,*

$$(j-1)k < a_j \leq jk \quad \text{and} \quad (j-1)k < b_j \leq jk. \tag{3}$$

Define the mapping $f : T_1 \to T_2$ by $f(u_{a_j}) = v_{b_j}$, for $j \in \{1, 2, \ldots, s\}$. Then the clique sum G of (H_1, T_1) and (H_2, T_2) through f satisfies $\chi_c(G) < t$.

Proof. For notational simplicity, let us identify u_{a_j} and v_{b_j} for each $j \in \{1, 2, \ldots, s\}$. From the symmetry, we may assume that $a_1 = b_1 = 1$. Also, let us define $n = tk - 1$. Consider the linear program $\mathcal{LP}_1$:

$$\begin{array}{llll}
r^* = \text{minimize } r & & & \\
\quad \text{s.t. } r + x_i - x_{i-k} \geq 1, & i \in \{1, 2, \ldots, k\} & [p_1, p_2, \ldots, p_k] \\
\quad x_i - x_{i-k} \geq 1, & i \in \{k+1, k+2, \ldots, n\} & [p_{k+1}, p_{k+2}, \ldots, p_n] \\
\quad r + y_i - y_{i-k} \geq 1, & i \in \{1, 2, \ldots, k\} & [q_1, q_2, \ldots, q_k] \\
\quad y_i - y_{i-k} \geq 1, & i \in \{k+1, k+2, \ldots, n\} & [q_{k+1}, q_{k+2}, \ldots, q_n] \\
\quad 0 = x_1 \leq x_2 \leq \ldots \leq x_n & & (*) \\
\quad 0 = y_1 \leq y_2 \leq \ldots \leq y_n & & (*) \\
\quad x_{a_i} = y_{b_i}, & i \in \{1, 2, \ldots, s\}. & [z_i]
\end{array}$$

(the variables in square brackets will denote dual variables, see below.)

Let $\mathcal{LP}_2$ be the program $\mathcal{LP}_1$ but with the constraints marked with $(*)$ dropped. We claim that in order to prove the Lemma, it suffices to show that the optimal value of $\mathcal{LP}_2$ is strictly smaller than t. For let $(r^*, \mathbf{x}, \mathbf{y}) = (r^*, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ be an optimal solution of $\mathcal{LP}_2$ with $r^* < t$. It is easy to check that $(r^*, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = (r^*, 0, x_2 - x_1, \ldots, x_n - x_1, 0, y_2 - y_1, \ldots, y_n - y_1)$ is also an optimal solution of $\mathcal{LP}_2$. Moreover, it follows from the fact that $r^* < t$ and the first four sets of constraints that $(r^*, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ satisfies the inequality constraints marked with $(*)$. From this and the fact that the feasible region of $\mathcal{LP}_1$ is a subset of the feasible region of $\mathcal{LP}_2$, it follows that

$(r^*, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is optimal for $\mathcal{LP}_1$. Now define the mapping $c : V(G) \to [0, r^*)$ by $c(u_i) = \tilde{x}_i$ and $c(v_i) = \tilde{y}_i$, $i \in \{1, 2, \ldots, n\}$. It is easy to check that c is a circular coloring of G and hence $\chi_c(G) \le r^* < t$.

In order to show this, consider the linear programming dual problem of $\mathcal{LP}_2$, with decision variables $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, z_1, z_2, \ldots, z_s$:

$$
\begin{aligned}
r^* = \max \; & \sum_{i=1}^{n} (p_i + q_i) \\
\text{s.t. } & p_i = p_{i+k}, && i \in \{1,2,\ldots,n\} \backslash \{a_1, a_2, \ldots, a_s\} && \text{(4a)} \\
& p_i = p_{i+k} - z_j, && i = a_j, j \in \{1,2,\ldots,s\} && \text{(4b)} \\
& q_i = q_{i+k}, && i \in \{1,2,\ldots,n\} \backslash \{b_1, b_2, \ldots, b_s\} && \text{(4c)} \\
& q_i = q_{i+k} + z_j, && i = b_j, j \in \{1,2,\ldots,s\} && \text{(4d)} \\
& \sum_{i=1}^{k} (p_i + q_i) = 1 && && \text{(4e)} \\
& p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \ge 0
\end{aligned}
$$

This dual can be interpreted as follows. Let us represent the vertices of H_1 by the following $t \times k$ "matrix" of vertices:

$$
M_u = \begin{bmatrix}
u_1 & u_2 & \cdots & u_{k-1} & u_k \\
u_{k+1} & u_{k+2} & \cdots & u_{2k-1} & u_{2k} \\
u_{2k+1} & u_{2k+2} & \cdots & u_{3k-1} & u_{3k} \\
\vdots & \vdots & & \vdots & \vdots \\
u_{(t-2)k+1} & u_{(t-2)k+2} & \cdots & u_{(t-1)k-1} & u_{(t-1)k} \\
u_{(t-1)k+1} & u_{(t-1)k+2} & \cdots & u_{tk-1} & \square
\end{bmatrix},
$$

where $\square$ denotes an "empty" entry. For $j \in \{1, 2, \ldots, s\}$, let r_j^A and c_j^A denote the row and column index, respectively, of vertex u_{a_j} in this matrix. Since we asserted that $a_1 = 1$, we have $c_1^A = r_1^A = 1$. It follows from (3) that $r_j^A = j$ for each $j \in \{1, 2, \ldots, s\}$, and from the definition of $K_{(tk-1)/k}$, it follows that $c_j^A \ge c_{j-1}^A$ for $j \in \{2, \ldots, s\}$.

For $j \in \{1, 2, \ldots, s\}$, consider a "walk" through the elements of M_u that starts at vertex u_{a_j} and moves down one row at a time, wrapping around to the next column and the first row whenever the bottom of the matrix is hit, until vertex $u_{a_{j+1}}$ is hit (where we let $a_{s+1} = 1$). Let A_j be the set of elements of M_u that are hit on this walk, except the starting vertex u_{a_j}, but including the final vertex $u_{a_{j+1}}$. Moreover, let $\bar{A}_j = A_j \cap \{u_1, u_2, \ldots, u_k\}$. It is clear from the matrix representation that $A_1, A_2, \ldots, A_s$ form a partition of $\{u_1, u_2, \ldots, u_n\}$. Also, it is easy to see that

$$
|\bar{A}_j| = c_{j+1}^A - c_j^A; \text{ and } |A_j| = |\bar{A}_j| t + 1. \tag{5}
$$

For every $i \in \{1, 2, \ldots, n\}$, the value of p_i in a feasible solution of the dual problem can be thought of as a weight assigned to vertex u_i. Constraints (4a) and (4b) state that equal weight is assigned to each vertex in A_j. That is, for each $j \in \{1, 2, \ldots, s\}$, $p_i = w_j^A$ for all $u_i \in A_j$ for some $w_j^A \in \mathbb{R}$.

We define M_v, B_j and $\bar{B}_j$ analogously to M_u, A_j and $\bar{A}_j$, but replacing the roles of a, u, r_j^A, c_j^A by b, v, r_j^B, c_j^B, respectively. Constraints (4c) and (4d) state the analogues of (4a) and (4b) for B_j, q_i and w_j^B. Constraint (4e) states that the sum of the weights p_i and q_i assigned to the vertices of the first row of M_u and M_v equals 1.

With this interpretation in mind, the dual problem can be rewritten as the following linear program with decision variables $w_1^A, w_2^A, \ldots, w_s^A, w_1^B, w_2^B, \ldots, w_s^B, z_1, z_2, \ldots, z_s$:

$$
\begin{aligned}
r^* = \max\ & \sum_{j=1}^{s} \big(|A_j| w_j^A + |B_j| w_j^B\big) \\
\text{s.t.}\ & \sum_{j=1}^{s} \big(|\bar{A}_j| w_j^A + |\bar{B}_j| w_j^B\big) = 1 \\
& w_j^A = w_{j+1}^A + z_j, && j \in \{1, 2, \ldots, s\} \\
& w_j^B = w_{j+1}^B - z_j, && j \in \{1, 2, \ldots, s\} \\
& w_j^A, w_j^B \geq 0, && j \in \{1, 2, \ldots, s\}.
\end{aligned}
$$

Using the facts that $\sum_{i=1}^{s} |A_j| = \sum_{i=1}^{s} |B_j| = n$ and $\sum_{i=1}^{s} |\bar{A}_j| = \sum_{i=1}^{s} |\bar{B}_j| = k$ and substituting the equality constraints into the objective function, this can be written as

$$
\begin{aligned}
r^* = \max\ & n(w_1^A + w_1^B) + \sum_{j=2}^{s} \big(|A_j| - |B_j|\big) z_j \\
\text{s.t.}\ & k(w_1^A + w_1^B) + \sum_{j=2}^{s} \big(|\bar{A}_j| - |\bar{B}_j|\big) z_j = 1 \\
& w_1^A, w_1^B \geq 0, \quad -w_1^A \leq \sum_{l=1}^{j} z_l \leq w_1^B, \quad j \in \{1, 2, \ldots, s\}.
\end{aligned}
\tag{6}
$$

It follows from (5) and its counterpart for H_2 that for $j \in \{1, 2, \ldots, s\}$, $|A_j| - |B_j| = t(|\bar{A}_j| - |\bar{B}_j|)$. Using this and the fact that $n = tk - 1$, the objective function can be written as

$$
tk(w_1^A + w_1^B) + t \sum_{j=2}^{s} \big(|\bar{A}_j| - |\bar{B}_j|\big) z_j - (w_1^A + w_1^B) \quad = \quad t - (w_1^A + w_1^B)
$$

Dropping the subscripts from w_1^A and w_1^B, the problem becomes

$$
\begin{aligned}
r^* = \text{maximize}\ & t - (w^A + w^B) \\
\text{s.t.}\ & k(w^A + w^B) + \sum_{j=2}^{s} \big(|\bar{A}_j| - |\bar{B}_j|\big) z_j = 1 \\
& w^A, w^B \geq 0, \quad -w^A \leq \sum_{l=1}^{j} z_l \leq w^B, \quad j \in \{1, 2, \ldots, s\}.
\end{aligned}
$$

It follows from linear programming duality that this problem is feasible (the primal problem has an obvious feasible solution). Clearly, since $w^A, w^B \geq 0$ for every feasible solution, we have $r^* \leq t$. Now suppose that there is a solution $(w^A, w^B, z_1, z_2, \ldots, z_s)$ that has objective value exactly t. This solution satisfies $w^A + w^B = 0$ and hence, by the inequality constraints $w^A \geq 0, w^B \geq 0$, it follows that $w^A = w^B = 0$. Consequently, it follows that $z_j = 0$ for all $j \in \{1, 2, \ldots, s\}$. This however contradicts the first constraint. Hence there exists no feasible solution that has objective value greater or equal to t and therefore $r^* < t$, which is what had to be proved. This completes the proof of Lemma 3.4. □

We need the following technical lemma.

Lemma 3.5. *Let $k \geq 1$ be an integer and let $s \in \{1, 2, 3\}$. Let G be a graph with $\chi_c(G) \leq 4 - 1/k$ and let $\{v_1, v_2, \ldots, v_{4k-1}\}$ denote the vertices of $K_{(4k-1)/k}$ such that $v_i v_j \in E(K_{(4k-1)/k}) \iff k \leq |i - j| \leq 3k - 1$. Suppose that $\{t_1, t_2, \ldots, t_s\}$ is a clique in G. Then there exists a homomorphism g from G to $K_{(4k-1)/k}$ and integers $a_1, \ldots, a_s$ such that $g(t_j) = v_{a_j}$ and $(j - 1)k < a_j \leq jk$, for every $j \in \{1, 2, \ldots, s\}$.*

Proof. By Theorem 3.2 and the choice of k, there exists a homomorphism g from G to $K_{(4k-1)/k}$. From the definition of a homomorphism, it follows that $g(\{t_1, \ldots, t_s\})$ is a clique in $K_{(4k-1)/k}$. From the symmetry, we may assume that $g(t_1) = v_1$. Let $a_1 = 1$. If $s = 1$, the lemma holds. Otherwise, let a_2 be such that $g(t_2) = v_{a_2}$. From the symmetry we may assume in addition that $a_2 \leq 2k$. It follows that $a_2 > k$. If $s = 2$, the lemma holds. Let a_3 be such that $g(t_3) = a_3$. If $s = 3$, since $t_2 t_3$ and $t_1 t_3$ are edges, it follows that $2k < a_3 \leq 3k$. This completes the proof of Lemma 3.5. □

We can now prove the main result of Subsection 3.2.

Lemma 3.6. *Let $s \in \{1, 2, 3\}$ and let H_1, H_2 be two K_4-free graphs. Let $T_1 \subseteq V(H_1)$ and $T_2 \subseteq V(H_2)$ be two cliques of size s, and let $f : T_1 \to T_2$ be a bijective mapping. Let G be the clique sum of (H_1, T_1) and (H_2, T_2) through f. If $\max\{\chi_c(H_1), \chi_c(H_2)\} < 4$, then $\chi_c(G) < 4$.*

Proof. Let k be an integer such that $\max\{\chi_c(H_1), \chi_c(H_2)\} \leq 4 - 1/k$ and let K_1 and K_2 be two disjoint a copies of $K_{(4k-1)/k}$ with vertex sets $\{u_1, u_2, \ldots, u_{4k-1}\}$ and $\{v_1, v_2, \ldots, v_{4k-1}\}$, respectively. Write $T_1 = \{r_1, r_2, \ldots, r_s\}$ and $T_2 = \{t_1, t_2, \ldots, t_s\}$ and assume without loss of generality that $f(r_j) = t_j$ for $j \in \{1, 2, \ldots, s\}$. From Lemma 3.5 it follows that there exist homomorphisms g_1, g_2 from H_1, H_2 to K_1, K_2, respectively, such that $g_1(r_j) = u_{a_j}$ and $g_2(t_j) = v_{b_j}$ satisfying $(j - 1)k < a_j \leq jk$ and $(j - 1)k < b_j \leq jk$ for $j \in \{1, 2, \ldots, s\}$.

Now consider the clique sum M of $(K_1, g(T_1))$ and $(K_2, g(T_2))$ through $g_2 \circ f \circ g_1^{-1}$. It follows from Lemma 3.4 that $\chi_c(M) < 4$ and hence there exists a homomorphism h from M to $K_{(4k'-1)/k'}$ for some integer $k' \geq 1$. Define the function $g : V(G) \to V(M)$ by $g(x) = g_1(x)$ if $x \in V(H_1)$ and $g(x) = g_2(x)$ if $x \in V(H_2) \backslash T_2$. It is easy to see that this is a homomorphism from G to M. Now $h \circ g$ is a homomorphism from G to $K_{(4k'-1)/k'}$ and hence $\chi_c(G) \leq \frac{4k'-1}{k'} < 4$ by Theorem 3.2. This proves Lemma 3.6. □

3.3 Circular coloring of K_4-free graphs with no odd hole

We are now in a position to prove our second main result:

Proof of Theorem 1.2. We prove this by induction on $|V(G)|$. Let G be a K_4-free graph with no odd hole. If G is perfect, then, by definition of a perfect graph, $\chi_c(G) \leq \chi(G) = \omega(G) \leq 3$ and hence the theorem holds. Therefore we may assume that G is not perfect. By Theorem 1.1, either G is isomorphic to T_{11} or $\bar{C}_7$, or G has an even pair, or G has a clique cutset. If G is isomorphic to T_{11} or $\bar{C}_7$, then $\chi_c(G) \in \{11/3, 7/2\}$, and hence theorem holds. If G contains an even pair $\{x, y\}$, then consider G/xy. It is easy to see that G/xy is K_4-free and has no odd hole. Hence from Lemma 3.1 and the induction hypothesis it follows that $\chi_c(G) \leq \chi_c(G/xy) < 4$. If G has a clique cutset X, then let C_1 be a connected component of $G \backslash X$, let $C_2 = V(G) \backslash C_1$, let $H_1 = G|(C_1 \cup X)$ and let $H_2 = G|C_2$. Clearly, H_1 and H_2 are K_4-free and do not have odd holes and therefore it follows from the induction hypothesis that $\chi_c(H_i) < 4$ for $i = 1, 2$. Since G is the clique sum of (H_1, X) and (H_2, X) through the identity function $f : X \to X$, it follows from Lemma 3.6 that $\chi_c(G) < 4$. This proves Theorem 1.2. □

We note that the requirement of forbidding odd holes is necessary, because if G is a triangle-free graph with $\chi(G) = 3$, then $\chi_c(M(G)) = 4$, where $M(G)$ denotes the Mycielskian of G (see [2]). Also, it is tempting to extrapolate the statement of this theorem and conjecture that, for every integer $p \geq 4$, if G is a K_p-free graph with no odd hole, then $\chi_c(G) < p$. However, this is already false for $p = 5$:

Claim 3.7. *Let G_1 and G_2 be two copies of $K_{14/3}$ with vertex sets $\{u_1, u_2, \ldots, u_{14}\}$ and $\{v_1, v_2, \ldots, v_{14}\}$, respectively, such that $u_i u_j \in E(G_1) \iff v_i v_j \in E(G_2) \iff 3 \leq |i - j| \leq 11$. Define $f : \{u_1, u_4\} \to \{v_1, v_7\}$ by $f(u_1) = v_1$ and $f(u_4) = v_7$. Then the clique sum G of G_1 and G_2 through f satisfies $\chi_c(G) = 5$.*

Proof. Since $\chi_c(G_1) = \chi_c(G_2) = 14/3$, it follows that $\chi(G_1) = \chi(G_2) = 5$. By vertex-coloring G_1 and G_2 separately and permuting the colors for G_2 so that the colors assigned to u_1 and v_1 and to u_4 and v_7 match, we can construct a 5-coloring of G by combining the colorings. It follows that $\chi_c(G) \leq \chi(G) \leq 5$. Now suppose that G has a circular r-coloring c with $r < 5$. We may assume that $c(v_1) = c(u_1) = 0$. From Lemma 3.3 and the symmetry, we may assume that $\{v_1, v_2, \ldots, v_{14}\}$ is circularly colored clockwise and $\{u_1, u_2, \ldots, u_{14}\}$ is either circularly colored clockwise or circularly colored counterclockwise. If $\{u_1, u_2, \ldots, u_{14}\}$ is circularly colored clockwise, since $u_1 u_4$ is an edge, it follows that $c(u_4) \geq 1$ and hence, since $u_4 u_7$ is an edge, that $c(v_4) = c(u_7) \geq 2$. If $\{u_1, u_2, \ldots, u_{14}\}$ is circularly colored counterclockwise, the same conclusion holds because $u_1 u_{11}$ and $u_{11} u_7$ are edges. But now from the fact that $v_4 v_7$ and $v_7 v_{10}$ are edges and because $\{v_1, v_2, \ldots, v_{14}\}$ is circularly colored clockwise, it follows that $c(v_{10}) \in [4, r)$. From the fact that $c(v_1) = 0$ and $v_1 v_{10}$ is an edge, it follows that $c(v_{10}) \in [1, r-1)$. But this is impossible because $r < 5$. This proves Claim 3.7. □

Finally, the bound in Theorem 1.2 is tight in the sense that there exists a sequence $\{H_n\}$ of K_4-free graphs with no odd hole such that $\chi_c(H_n) \to 4$ as $n \to \infty$. This sequence can be constructed as follows. Let H_1 be a copy of $\bar{C}_7$ with vertex set $\{v_1^1, v_2^1, \ldots, v_7^1\}$. For $k \geq 2$, let G_k be a copy of $\bar{C}_7$ with vertex set $\{v_1^k, v_2^k, \ldots, v_7^k\}$, let

$T_1 = \{v_2^{k-1}, v_7^{k-1}\}$, let $T_2 = \{v_6^k, v_3^k\}$, and let $f : T_1 \to T_2$ be defined by $f(v_2^{k-1}) = v_6^k$ and $f(v_7^{k-1}) = v_3^k$. We define H_k as the clique sum of (H_{k-1}, T_1) and (G_k, T_2) through f. (See Figure 3) We have the following result.

Theorem 3.8. *For every $n \geq 1$, $\chi_c(H_n) \geq 4 - \frac{1}{n+1}$.*

Proof. Let $n \geq 1$ be given. Let $c : V(H_n) \to [0, r)$ be an optimal circular coloring of H_n. It follows from Lemma 3.6 that $r = 4 - \varepsilon$ for some $\varepsilon > 0$. From the symmetry, we may assume that $c(v_1^1) = 0$ and that $c(v_2^1) < r/2$. From this and from Lemma 3.3, it follows that $v_1^1, v_2^1, \ldots, v_7^1$ are circularly colored clockwise. Let us first prove the following:

(∗) *Let $1 \leq k \leq n$. If $k = 2p+1$ for some integer $p \geq 0$, then*

$$(1+p)\varepsilon \leq c(v_2^k) \leq 1 - (1+p)\varepsilon \quad \text{and} \quad 3 + p\varepsilon \leq c(v_7^k) \leq 4 - (2+p)\varepsilon.$$

If $k = 2p$ for some integer $p \geq 1$, then

$$2 + p\varepsilon \leq c(v_2^k) \leq 3 - (1+p)\varepsilon \quad \text{and} \quad 1 + p\varepsilon \leq c(v_7^k) \leq 2 - (1+p)\varepsilon.$$

We will prove this by induction on k. First suppose that $k = 1$. Recall that $v_1^1, v_2^1, \ldots, v_7^1$ are circularly colored clockwise. Hence, since v_1^1-v_3^1-v_5^1-v_7^1-v_2^1 and v_1^1-v_6^1-v_4^1-v_2^1-v_7^1 are paths, it follows that $3 \leq c(v_7^1) \leq 4 - 2\varepsilon$ and $\varepsilon \leq c(v_2^1) \leq 1 - \varepsilon$. This proves (∗) for $k = 1$.

Next suppose that $k = 2p+1$ for some integer $p \geq 1$. It follows from the induction hypothesis that $c(v_3^k) = c(v_7^{k-1}) \geq 1 + p\varepsilon$ and $c(v_6^k) = c(v_2^{k-1}) \leq 3 - (1+p)\varepsilon$. We first claim that $v_1^k, v_2^k, \ldots, v_7^k$ are circularly colored clockwise. For suppose otherwise. Then by Lemma 3.3, $v_1^k, v_2^k, \ldots, v_7^k$ are circularly colored counterclockwise. This, together with the fact that v_3^k-v_1^k-v_6^k is a path, implies that $c(v_6^k) \geq 3 + p\varepsilon$, contrary to the fact that $c(v_6^k) \leq 3 - (1+p)\varepsilon$. So $v_1^k, v_2^k, \ldots, v_7^k$ are circularly colored clockwise. Hence, since v_3^k-v_5^k-v_7^k-v_2^k and v_6^k-v_4^k-v_2^k-v_7^k are paths, it follows that $3 + p\varepsilon \leq c(v_7^k) \leq 4 - (2+p)\varepsilon$ and $(1+p)\varepsilon \leq c(v_2^k) \leq 1 - (1+p)\varepsilon$. This proves (∗) for odd k.

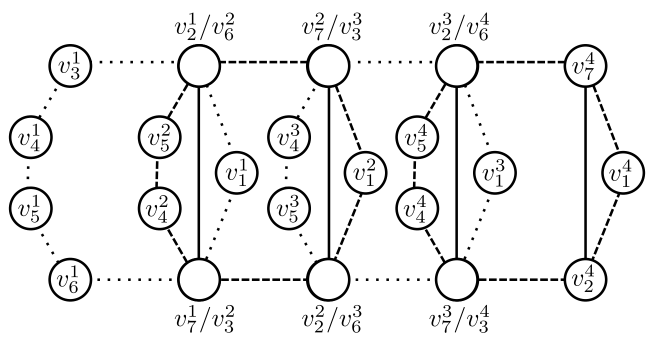

Figure 3: The graph H_4. In this figure, every heptagon represents a copy of $\bar{C}_7$ and the dotted lines represents non-edges.

Finally suppose that $k = 2p$ for some integer $p \geq 1$. It follows from the induction hypothesis that $c(v_3^k) = c(v_7^{k-1}) \geq 3 + (p-1)\varepsilon$ and $c(v_6^k) = c(v_2^{k-1}) \leq 1 - p\varepsilon$. We first claim that $v_1^k, v_2^k, \ldots, v_7^k$ are circularly colored clockwise. For suppose otherwise. Then by Lemma 3.3, $v_1^k, v_2^k, \ldots, v_7^k$ are circularly colored counterclockwise. This, together with the fact that v_3^k-v_1^k-v_6^k is a path, implies that $c(v_6^k) \geq 1 + p\varepsilon$, contrary to the earlier observation that $c(v_6^k) \leq 1 - p\varepsilon$. So $v_1^k, v_2^k, \ldots, v_7^k$ are circularly colored clockwise. Hence, since v_3^k-v_5^k-v_7^k-v_2^k and v_6^k-v_4^k-v_2^k-v_7^k are paths, it follows that $1 + p\varepsilon \leq c(v_7^k) \leq 2 - (1+p)\varepsilon$. and $2 + p\varepsilon \leq c(v_2^k) \leq 3 - (1+p)\varepsilon$. This completes the proof of $(*)$.

We can now prove the theorem. If $n = 2p + 1$ for some integer $p \geq 0$, it follows from $(*)$ that $(1+p)\varepsilon \leq 1 - (1+p)\varepsilon$, which is equivalent to $\varepsilon \leq \frac{1}{2(p+1)} = \frac{1}{n+1}$. If $n = 2p$ for some integer $p \geq 1$, then it follows from $(*)$ that $2 + p\varepsilon \leq 3 - (1+p)\varepsilon$, which is equivalent to $\varepsilon \leq \frac{1}{1+2p} = \frac{1}{n+1}$. Finally, $\varepsilon \leq \frac{1}{n+1}$ implies that $\chi_c(H_n) \geq 4 - \frac{1}{n+1}$, completing the proof of Theorem 3.8. □

Note that using the techniques of Lemma 3.4, it can be shown that in fact $\chi_c(H_n) = 4 - 1/(n+1)$.

4 Conclusion

In this paper we used the structural theorem presented in [3] to obtain a structural result on even pairs in K_4-free graphs with no odd holes. We used the new result in combination with linear programming duality theory to show that every such K_4-free graph with no odd hole has circular chromatic number strictly less than 4. We showed that this result is tight by giving an explicit construction for graphs whose circular chromatic number is arbitrarily close to 4. We indicated that it seems natural to conjecture that every K_p-free ($p \geq 4$) graph with no odd hole has circular chromatic number strictly less than p. However, this is false.

Acknowledgements I would like to thank Maria Chudnovsky and the anonymous referees for numerous helpful comments and suggestions.

References

[1] J.A. Bondy and P. Hell. A note on the star chromatic number. *J. Graph Theory*, 14(4):479–482, 1990.

[2] G.J. Chang, L. Huang, and X. Zhu. The circular chromatic number of Mycielski's graphs. *Discrete Math*, 205:23–37, 1999.

[3] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. K_4-free graphs with no odd holes. *To appear in Journal of Combinatorial Theory B*, 2008.

[4] J. Fonlupt and JP Uhry. Transformations which preserve perfectness and h-perfectness of graphs. *Ann. Disc. Math*, 16:83–85, 1982.

[5] J.L. Ramírez-Alfonsín and B.A. Reed. *Perfect Graphs*. Wiley Interscience, New York, 2001.

[6] A. Tucker. Coloring graphs with stable cutsets. *J. Combinatorial Theory, Series B*, 34(3):258–267, 1983.

[7] A. Vince. Star chromatic number. *J. Graph Theory*, 12(4):551–559, 1988.

[8] X. Zhu. Circular chromatic number: a survey. *Discrete Math*, 229(1-3):371–410, 2001.

[9] X. Zhu. Recent developments in circular colorings of graphs. *Topics Discrete Math*, pages 497–550, 2006.