Forbidding Induced Subgraphs

Yori Zwols

Submitted in partial fulfillment of the

Requirements for the degree

of Doctor of Philosophy

in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2010

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ABSTRACT

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A graph H is called an *induced subgraph* of G if H can be obtained from G by repeatedly deleting a vertex and all edges incident with it. The theory of forbidden induced subgraphs deals with classes of graphs that are closed under taking induced subgraphs. These classes can equivalently be characterized by forbidding certain fixed graphs to appear as an induced subgraph.

In this thesis, we will present new results on several classes of graphs that are closed under taking induced subgraphs. We have two results that are related to two famous conjectures in the field, and we will characterize a class of graphs which naturally arises in a wireless communication network application in electrical engineering.

The first result is related to a conjecture of Gyárfás **[33]** that states that the chromatic number of a graph that has no induced cycles of odd length at least five is bounded by a function of the size of its largest complete subgraph. We will present a result on the so-called circular chromatic number (a refinement of the usual vertex chromatic number) of such graphs when the clique number is at most three. For this, we use a structural description of such graphs, in conjunction with a linear programming duality argument.

The second result is related to a conjecture of Erdős and Hajnal **[26]** that has recently become quite popular among the researchers in the field. This conjecture states that for every fixed graph H, there exists a constant $c > 0$ such that every graph that does not contain H as an induced subgraph has a clique (*i.e.*, a set of pairwise adjacent vertices) or stable set (*i.e.*, a set of pairwise nonadjacent vertices) of size at least $|V(G)|^c$. One of the many open cases for this conjecture is the case when H is a four-edge path. We present a partial result for the case when H is the four-edge path.

The third result is motivated by a scheduling problem in wireless communication networking. Wireless communication is more complicated than communication in wired networks because connections between transmitters and receivers may interfere with each other. As a result, optimal scheduling algorithms for wireless communication generally require centralized computing, which is often not desirable. In **[23]**, sufficient conditions were given that ensure 'optimality' for a distributed scheduling algorithm (this optimality will be made precise). These conditions can be interpreted as a graph-theoretical property on the graph that represents a wireless network. We characterize completely the set of 'line graphs' for which these conditions are met. Line graphs form a natural class of graphs

in the wireless communication setting because they coincide with one of the simplest conceivable interference models. We then extend this characterization to a more general class of graphs, the so-called 'claw-free graphs', and obtain interesting graph theoretic properties of the claw-free graphs that satisfy the conditions given in [23].

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Acknowledgments

After visiting the Industrial Engineering and Operations Research department at Columbia University as an incoming Ph.D. student in the Spring of 2006, I remember receiving an email from the director of graduate studies, Prof. Ward Whitt. He wrote me to tell me that a certain Maria Chudnovsky would be joining the IEOR department in the following semester. Although I was determined to work on probabilistic problems, he also urged me to take Maria's class in graph theory. As you can imagine after reading the title of this thesis, this turned out to be a good idea.

I can now hardly imagine that I was once scared to walk into Maria's office to ask if I could work with her during the summer: she is one of the friendliest persons I have ever met. Her patience and dedication to being both an excellent researcher and a great supervisor have amazed me time and time again in these past few years. So, Maria: thank you very much for all your time and support. I am proud to have been your student, and I sincerely hope that we will continue working together in the future.

Another person that has greatly influenced my academic life is Gerard Sierksma. Gerard was my Master's thesis adviser at the University of Groningen, but, much more importantly, is now a great friend and almost a second father to me. Gerard, I want to thank you for your ever-lasting support and confidence in me.

To my friends at Columbia – Ceci, Romain, Bernard, Matthieu, Tulia, Rodrigo, Serhat and all others: thank you so much for making my time at Columbia a great experience. We will surely meet again.

Next, this thesis and even the opportunity of writing it in the United States, would never have been possible without my family: my parents Johan and Jeannet, my sister Jiddy, and my brother Josse. In particular my parents, whose caring from my childhood on has been crucial in getting to where I am right now, and their eternal support does not go unnoticed. And although I am geographically far away from you, I think that spending quality time together, both in New York and in Groningen, has brought us closer together. I love you very much.

And last but of course far from least: Rouba, without all the love you give me my life would have been unimaginably different. We'll conquer Montreal and other places to come together.

Yori Zwols
New York, June 20th 2010

Definitions and notation

This section contains notation and definitions that will be used throughout the thesis. It is included for convenience: the less obvious definitions will be repeated within the text whenever they are needed, so that the reader may skip this section at first reading.

All graphs in this thesis are finite and simple, *i.e.*, they do not have parallel edges or loops. In what follows, let G be a graph.

Basic graph notions We denote by $V(G)$ the vertex set of G, and by $E(G)$ the edge set of G. For $v \in V(G)$, we say that $u \in V(G) \setminus \{v\}$ is a *neighbor* (*nonneighbor*) of v if u is adjacent (nonadjacent) to v. We denote by $N_G(v)$ and $M_G(v)$ the set of neighbors and nonneighbors, respectively, of v in G. $N_G(v)$ is called the *open neighborhood of* v. Let $N_G[v] = N_G(v) \cup \{v\}$. We call $N_G[v]$ the *closed neighborhood of* v. For a set $X \subseteq V(G)$, let $N_G(X) = \left(\bigcup_{v \in X} N_G(v)\right) \setminus X$ and $N_G[X] = N_G(X) \cup X$. The *degree of* v *in* G is denoted by $\deg_G(v)$ and is equal to $|N_G(v)|$. In most cases, when it is clear from the context what G is, we drop the subscript G.

Let $X, Y \subseteq V(G)$ be two disjoint sets. We say that X is *complete* (*anticomplete*) *to* Y if every vertex in X is adjacent (nonadjacent) to every vertex in Y. Clearly, X is complete (anticomplete) to Y if and only if Y is complete (anticomplete) to X, and hence we may say that X and Y are *complete* (*anticomplete*) *to each other*. We say that X and Y are *linked* if every vertex of X has a neighbor in Y and every vertex of Y has a neighbor in X.

Let G^c be the graph with vertex set $V(G^c) = V(G)$ and, for distinct $u, v \in V(G^c)$, $uv \in E(G^c)$ if and only if $uv \notin E(G)$. The graph G^c is called the *complement* of G. For $X \subseteq V(G)$ we denote by $G \setminus X$ the graph obtained from G by deleting all vertices in X and all edges that are incident with at least one vertex in X. For $x \in V(G)$, we let $G \setminus x = G \setminus \{x\}$. Let $u, v \in V(G)$ be distinct. Construct a new graph G/uv from G by deleting u, v and adding a new vertex uv that is adjacent to precisely the vertices that are adjacent to one of u, v in G (or both). We say that G/uv is obtained from G *by contracting the pair* $\{u, v\}$.

A set $X \subseteq V(G)$ is a *clique* (*stable set*) if the vertices in X are pairwise adjacent (nonadjacent). A clique (stable set) X is said to be *maximal* (*under inclusion*) in G if there is no $v \in V(G) \setminus X$ such that $X \cup \{v\}$ is a clique (stable set). Notice that a maximal clique should not be confused with a maximum (cardinality) clique.

A set $M \subseteq E(G)$ is called a *matching* if no two edges in M share an endpoint. A matching M is *maximal* if there is no $e \in E(G) \setminus M$ such that $M \cup \{e\}$ is a matching. A matching M is said to *cover* a vertex v, if there exists an edge in M that is incident with v.

Special cases of graphs Let $t, n \geq 1$. Let K_n denote a complete graph on n vertices and let $K_{n,t}$ denote a $n \times t$ complete bipartite graph. We denote by $K_{2,t}^{+}$ the graph that is the union of a clique X with $|X| = 2$ and a stable set Y with $|Y| = t$ such that X is complete to Y. We denote by C_n a cycle of length n, and by $\bar{C}_n$ the complement of C_n. We denote by P_n a path with n edges (and thus, with $n+1$ vertices).

Relationships between graphs For two graphs G_1, G_2, an *isomorphism from G_1 to G_2* is a bijection $\phi : V(G_1) \to V(G_2)$ such that for distinct $u, v \in V(G_1)$, u and v are adjacent in G_1 if and only if $\phi(u)$ and $\phi(v)$ are adjacent in G_2. For two graphs G_1, G_2, if there exists an isomorphism from G_1 to G_2, then G_1 and G_2 are *isomorphic*. An *automorphism of G* is an isomorphism from G to itself. For two graphs G_1, G_2, a *homomorphism from G_1 to G_2* is a function $\psi : V(G_1) \to V(G_2)$ such that $\psi(u)\psi(v) \in E(G_2)$ whenever $uv \in E(G_1)$.

A graph H is called a *minor of G* if H can be obtained from G by repeatedly deleting an edge, or a vertex and all edges incident with it, or contracting a pair of adjacent vertices. A graph H is called a *subgraph of G* if H can be obtained from G by repeatedly deleting an edge or a vertex and all edges incident with it. A graph H is called an *induced subgraph of G* if H can be obtained from G by repeatedly deleting a vertex and all edges incident with it. We say that *G contains H as a minor, subgraph, induced subgraph* if G has a minor, subgraph, induced subgraph, respectively, that is isomorphic to H.

A *subdivision* of G is a graph that can be obtained from G by repeatedly replacing edges by induced paths (this operation is called *subdividing an edge*).

Graph properties For $k \geq 1$, we say that G is *k-connected* if for every two distinct vertices in G, there exist k vertex-disjoint paths between them. As a special case, a graph is *connected* if it is 1-connected. A *connected component of G* is a maximal connected subgraph of G. A graph is *disconnected* if it is not connected. We denote by $\alpha(G)$ and $\omega(G)$ the size of the largest stable set and the largest clique, respectively, in G.

For an integer $k \geq 1$, a *k-vertex-coloring of G* is a function $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ whenever $u, v \in V(G)$ are adjacent. We denote by $\chi(G)$ the smallest $k \geq 1$ such that G has a k-vertex-coloring. We call $\chi(G)$ the *chromatic number* of G.

For a real number $r > 0$ and a graph G, a *circular r-coloring of G* is a function $c : V(G) \to [0, r)$ such that $1 \leq |c(u) - c(v)| \leq r - 1$ whenever $uv \in E(G)$. The *circular chromatic number of G*, denoted $\chi_c(G)$, is defined as $\chi_c(G) = \inf\{r : G \text{ has a circular } r\text{-coloring}\}$.

Structures in graphs An *induced path* in G is an induced subgraph of G that is a path. An induced path is called *even* if it has an even number of edges, and *odd* if it has an odd number of edges. An *even pair* is a pair $\{u, v\}$ of vertices of G such that all induced paths in G from u to v are even. An *induced cycle* or a *hole* is an induced subgraph of G that is a cycle of length at least four. An *antihole* in a graph G is an induced subgraph of G that is the complement of a cycle of length at least four. A hole (or antihole) is said to be *even* or *odd* if it has even or odd length, respectively. A *cutset* in a connected graph G is a set $X \subseteq V(G)$ such that $G \backslash X$ is disconnected. A *clique cutset* is a cutset that is a clique.

Graph classes For graphs $H_1, H_2, \dots, H_k$, let $\mathrm{Forb}(H_1, H_2, \dots, H_k)$ be the set of all graphs G such that for all $i \in \{1, 2, \dots, k\}$, no induced subgraph of G is isomorphic to H_i. Given a graph H, the *line graph of H*, denoted $L(H)$, is a graph with vertex set $V(L(H)) = E(H)$ such that two distinct $e_1, e_2 \in V(L(H))$ are adjacent in $L(H)$ if and only if e_1 and e_2 share a vertex in H. A graph is called a *line graph* if it is the line graph of some graph. A graph G is called *claw-free* if G does not contain $K_{1,3}$ as an induced subgraph.

A graph G is called *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$, and G is called *imperfect* otherwise. A clique (stable set) X in G is called *dominant* if $X \cap Y \neq \emptyset$ for every maximal stable set (clique) Y in G. A graph G is *strongly perfect* if every induced subgraph G' of G has a dominant stable set. A graph G is *fractionally strongly perfect* if for every induced subgraph G' of G there exists a function $g : V(G') \to [0, 1]$ such that $\sum_{v \in K} g(v) = 1$ for every maximal clique K in G'. A graph G is *fractionally co-strongly perfect* if G^c is fractionally strongly perfect.

Proofs Many of the proofs in this thesis consist of one or more subclaims that are enumerated as **(i)**, **(ii)**, ... To increase the readibility of this thesis, a small open square ($\square$) signifies the end of the proof of such a subclaim, whereas the end of the proofs of 'outer' theorems and lemmas are signified by a larger closed square ($\blacksquare$).

List of notation used

$[n]$	the set $\{1, 2, \dots, n\}$
$V(G)$	the vertex set of G
$E(G)$	the edge set of G
G^c	the complement of G
$\deg_G(v)$	the degree of vertex v in G
$N(v)$	the set of vertices that are adjacent to v
$M(v)$	the set of vertices that are nonadjacent to v
$N[v]$	the set of v and all vertices that are adjacent to v
$G\|X$	the subgraph of G induced by $X \subseteq V(G)$
$\alpha(G)$	size of the largest stable set in G
$\omega(G)$	size of the largest clique in G
$\chi(G)$	the chromatic number of G
$\chi_c(G)$	the circular chromatic number of G
$\mathrm{Forb}(H_1, H_2, \dots, H_k)$	the set of all graphs that contain none of $H_1, H_2, \dots, H_k$ as induced subgraph

1 Introduction

If I had to choose a favorite theorem in graph theory, it would probably be Ramsey's theorem **[47]**. This theorem deals with what is sometimes called the *party problem*: what is the minimum number $R(s, t)$ of guests that we need to invite to a party so that there is either a group of s guests that all mutually know each other, or a group of t guests that are all mutually unfamiliar with each other? Ramsey's theorem states that $R(s, t)$ is finite for all s, t.

By interpreting guests as vertices of a graph and the 'knowing each other' relationship as its edges, the party problem translates directly into the following graph problem: what is the minimum number of vertices that are required to ensure that there is either a set of s vertices that are all pairwise adjacent, or a set of t vertices that are all pairwise nonadjacent? In this setting, Ramsey's theorem reads as follows.[1]

Theorem 1.0.1. (Ramsey's Theorem **[47]**) *For all $s, t \geq 1$, there exists $R(s, t)$ such that every graph on at least $R(s, t)$ vertices either has a clique of size s, or a stable set of size t.*

Ramsey's theorem deals with structures that are necessarily present in every large enough graph. This thesis deals with substructures in graphs, but from a different perspective: we are interested in classes of graphs in which certain fixed structures are *not* present.

The statement that a 'structure is present' in a graph leaves some space for interpretation. To make the interpretation precise, we need to specify a *containment relationship*. One common relationship is the *subgraph relationship*. A textbook definition for the subgraph relationship is (see for example **[31]** or **[58]**): a graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In that case, we say that G contains H as a subgraph.

An alternative way of defining the subgraph containment relationship is by specifying operations that

[1] In this chapter, we assume basic knowledge of graph theory. We will use standard definitions, but in case of doubt we refer the reader to the section "Definitions and notation" that starts on page ix.

we may perform on G such that after repeatedly performing these operations, we end up with a graph H that we say is a subgraph of G. For the subgraph relationship, these operations are: (1) deleting an edge and (2) deleting a vertex and all edges incident with it. In particular, an equivalent definition of a subgraph is the following:

Definition. *A graph H is a* subgraph *of G if H can be obtained from G by repeatedly (1) deleting an edge or (2) deleting a vertex and all edges incident with it. (We say that a graph G* contains H as a subgraph *if G has a subgraph that is isomorphic to H.)*

Over the past few decades, a new and exciting area in graph theory has emerged which deals with a containment relationship that is known as a 'graph minor'. This containment relationship is identical to the subgraph relationship, but we are allowed to perform one more operation, namely deleting two adjacent vertices u and v and all edges incident with them, and adding a new vertex w that is adjacent to precisely the vertices that were originally adjacent to at least one of u, v. This operation is called *contracting the pair of adjacent vertices* $\{u, v\}$. Thus, the definition of a minor of a graph reads as follows:

Definition. *A graph H is a* minor *of G if H can be obtained from G by repeatedly (1) deleting an edge, (2) deleting a vertex and all edges incident with it, or (3) contracting a pair of adjacent vertices. (We say that a graph G* contains H as a minor *if G has a minor that is isomorphic to H.)*

The theory of graph minors deals with classes of graphs that are closed under taking minors. Such classes can be characterized by *excluding* (or *forbidding*) a list of fixed graphs as minors. For example, an important minor-closed class of graphs is the class of *planar graphs*. These are the graphs that can be drawn in the plane such that no two edges cross one another. It is a theorem of Wagner **[56]** that a graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a minor. (More generally, for every fixed surface S, there exists a finite list of graphs $\mathcal{G}(S)$ such that a graph G may be embedded into the surface S if and only if G does not contain any of the graphs in $\mathcal{G}(S)$ as a minor.)

The theory of graph minors is quite developed, not in the least place due to the celebrated work of Neil Robertson and Paul Seymour. One of the main results of this work is the Graph Minor Theorem, which "may doubtlessly be counted among the deepest theorems that mathematics has to offer" **[22]**. Its proof spans over 500 pages distributed over around 20 papers. The Graph Minor Theorem states that graphs are *well-quasi-ordered* under the minor relationship: every infinite sequence of graphs contains two graphs such that one contains the other as a minor. Although this statement seems quite innocent, it has important implications in theoretical computer science and, more specifically, in complexity theory.

To wit, one of its implications is that every minor-closed class of graphs can be characterized by excluding a finite set of minors. Together with a theorem that states that one can determine in $O(|V(G)|^3)$ time **[48]** whether a given graph is a minor of another graph G, this implies that, for a minor-closed class of graphs $\mathcal{G}$, determining whether a given graph G is a member of $\mathcal{G}$ can be done in polynomial time.

1.1 Forbidden induced subgraph theory

In this thesis, we are interested in a different graph containment relationship, namely the *induced subgraph relationship*, which is defined as follows:

Definition. *A graph H is an* induced subgraph *of G if H can be obtained from G by repeatedly deleting a vertex and all edges incident with it. (We say that a graph G* contains H as an induced subgraph *if G has an induced subgraph that is isomorphic to H.) We denote by* $\text{Forb}(H_1, H_2, \ldots, H_k)$ *the class of graphs that do not contain any of $H_1, H_2, \ldots, H_k$ as an induced subgraph.*

In contrast to the case of graph minors, there exists no general theory of induced subgraphs, despite many research projects devoted to this relationship, including the work of many prominent researchers. A question one may ask is whether graphs are well-quasi-ordered under the induced subgraph relationship. This, however, is false: consider for example the sequence of cycles of length $3, 4, 5, \ldots$. It is clear that no cycle of length $k \geq 3$ contains another cycle of length $j \geq 3$, $j \neq k$, as an induced subgraph. Therefore, it is not true that every class of graphs that is closed under taking induced subgraphs can be characterized by excluding a finite collection of induced subgraphs.

An important example of an induced-subgraph-closed class of graphs is formed by the perfect graphs. For a graph G, denote by $\chi(G)$ its chromatic number, and by $\omega(G)$ the size of a largest clique in $\omega(G)$. A graph G is *perfect* if and only if $\chi(H) = \omega(H)$ for every induced subgraph H of G. One of the important results in forbidden induced subgraph theory is the Strong Perfect Graph Theorem which characterizes the class of perfect graphs in terms of its forbidden induced subgraphs. Before we state the theorem, let us say that a *hole* in a graph is an induced subgraph that is a cycle, and an *antihole* in a graph is an induced subgraph that is the complement of a cycle. A hole (antihole) is *odd* if it has an odd number of vertices. The following theorem was conjectured by Berge in 1963 **[4]**, and proved in 2002 by Chudnovsky, Seymour, Robertson and Thomas:

Theorem 1.1.1. (The Strong Perfect Graph Theorem **[17]**) *A graph G is perfect if and only if G contains no odd hole of length at least 5 and no odd antihole of length at least 5.*

In addition to applications in graph theory, Theorem 1.1.1 has important implications in theoretical and applied computer science, including, but not limited to, polyhedral theory, linear (integer) programming, and machine learning.

Most of the recent work in induced subgraph theory has focused on studying specific classes of graphs by specifying a fixed set of graphs that are to be excluded. Examples include perfect graphs, claw-free graphs, bull-free graphs, odd-hole-free graphs, even-hole-free graphs, and many more. Many important results on such specific classes of graphs have been shown by means of a structural description of graphs in these classes. Such structural descriptions are generally of the form: every graph in the class under consideration is either of a certain (hopefully well-understood) basic type, or can be decomposed into smaller parts. Such theorems are called *decomposition theorems*. For example, the Strong Perfect Graph Theorem was shown using this paradigm.

In some cases (*e.g.*, claw-free graphs), it is possible to find decompositions that are 'invertible', in the sense that the decomposition prescribes how a collection of graphs in the class of interest can be combined into a larger graph that is also in this class. Together with the decomposition theorem, this gives a complete description of how every graph in the class may be constructed from smaller pieces. We may therefore speak of a *structure theorem*, rather than 'just' a decomposition theorem.

Although many different *specific* classes of graphs that are characterized by their forbidden induced subgraphs have been studied in the literature, there are only few big *general* results in the field. However, there are many conjectures that point out what such general results may look like. In the next two sections, we discuss a few of these conjectures.

1.2 χ-Bounded classes of graphs

There are a few conjectures surrounding the chromatic number of graphs as a function of the clique number. For a graph G, the clique number $\omega(G)$ is a trivial lower bound for the chromatic number $\chi(G)$ of G. One may ask the question: can we bound $\chi(G)$ from above with some function of $\omega(G)$? The answer is 'no', as shown by the following theorem by Paul Erdős:

Theorem. [25] *For every $k \geq 1, g \geq 1$, there exists a graph G such that $\omega(G) = 2$, $\chi(G) \geq k$, and G has no cycle of length at most g.*

Thus, if we want to bound the chromatic number from above by a function of the clique number, we better concentrate on a smaller class of graphs than the class of all graphs.

Definition. *Let $\mathcal{F}$ be a collection of graphs. The function $f : \mathbb{N} \to \mathbb{N}$ is called a χ-*bounding function *for $\mathcal{F}$ if $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{F}$. If $\mathcal{F}$ has a χ-bounding function, then $\mathcal{F}$ is said to be* χ-bounded. *A χ-bounding function f is* optimal *for $\mathcal{F}$ if $f(k) \leq f'(k)$ for every χ-bounding function f' for $\mathcal{F}$.*

Observe that for a finite collection $\mathcal{F}$ of graphs, the function f such that, for all k, $f(k)$ equals the number of vertices in a largest graph in $\mathcal{F}$ is a χ-bounding function. Therefore, every finite collection of graphs is χ-bounded. So χ-boundedness is only interesting for infinite families of graphs. We note that, because $\chi(G) \geq \omega(G)$ for all graphs G, we have $f(k) \geq k$ for any χ-bounding function f.

One important class of graphs that is χ-bounded is the class of perfect graphs, for which the identity function $f(k) = k$ is the optimal χ-bounding function. As said, due to Theorem 1.1.1, perfect graphs are characterized by excluding the odd cycles of length at least five and their complements as induced subgraphs. Gyárfás conjectured that excluding only the odd cycles of length at least five as induced subgraphs suffices for obtaining a χ-bounded class. We refer to this class of graphs as the *odd-hole-free graphs*.

Conjecture 1.2.1. [33] *The class of odd-hole-free graphs is χ-bounded.*

Despite many efforts and the existence of a decomposition theorem for odd-hole-free graphs **[21]**, the conjecture is still open. A theorem that seems to go into the direction of the conjecture is a theorem by Scott, who proved that if, in addition to excluding odd holes, we exclude long even holes, then we obtain a χ-bounded class of graphs. To be precise, he proved the following theorem:

Theorem. [50] *For any $k \geq 1$, the class of graphs with no odd holes and no even holes of length at least k is χ-bounded.*

Although this theorem seems to get close to proving Conjecture 1.2.1, no-one has been able to extend it into a proof of Conjecture 1.2.1. Interestingly, it is known that the even-hole-free graphs (*i.e.*, the graphs that have no even hole of length at least four) form a χ-bounded class:

Theorem. [1] *Every even-hole-free graph G satisfies $\chi(G) \leq 2\omega(G) - 1$.*

Another beautiful conjecture made by Gyárfás (and later, independently, by Sumner) is the following:

Conjecture 1.2.2. [32, 52] *For every fixed tree T,* Forb(T) *is χ-bounded.*

Gyárfás **[33]** later proved the conjecture for the case when T is a path. Significant progress was made by Kierstead and Penrice **[38]** who proved it for the case when T is a tree of radius two. Kierstead and Zhu **[39]** extended this result to a certain class of radius-three trees. The only general result in the direction of Conjecture 1.2.2 is (again) due to Scott, who proved a version of it for a 'relaxed' version of the induced subgraph relationship:

Definition. *Let G be a graph. For two adjacent vertices $u, v \in V(G)$, let G' be the graph obtained from G by making u and v nonadjacent and adding a new vertex w that is adjacent to precisely u and v. We say that G' is obtained from G by* subdividing the edge uv. *A* subdivision of G *is a graph that can be obtained from G by repeatedly subdividing an edge. We say that a graph G* contains a subdivision of H *if G has an induced subgraph that is isomorphic to a subdivision of H.*[2]

Denote by $\text{Forb}^*(H)$ the class of graphs that contain no subdivision of H. In this setting, Scott proved the following:

Theorem. [51] $\text{Forb}^*(T)$ *is χ-bounded for every tree T.*

In the same paper, he made the following conjecture:

Conjecture. [51] $\text{Forb}^*(H)$ *is χ-bounded for every graph H.*

(One may also wonder: perhaps it is true that graphs are well-quasi-ordered under the induced subdivision relationship? The answer is 'no': consider the sequence of graphs that are constructed from a cycle of length $3, 4, 5, \ldots$ by adding one vertex that is adjacent to all vertices in the cycle.)

[2] In the setting of operations, an equivalent definition is the following: G contains a subdivision of H if H can be obtained from G by repeatedly (1) deleting a vertex and all edges incident with it, or (2) contracting an edge that is incident with a vertex of degree two.

1.2.1 Contributions

Our result deals with odd-hole-free graphs with clique number at most three, or equivalently: K_4-free graphs with no odd hole, where K_4 denotes the complete graph on 4 vertices. The following theorem of Chudnovsky, Robertson, Seymour and Thomas establishes that the chromatic number of such graphs is bounded:

Theorem 1.2.3. [20] *Every odd-hole-free graph G and $\omega(G) \leq 3$ satisfies $\chi(G) \leq 4$. Moreover, there exists an odd-hole-free graph G' with $\chi(G') = 4$.*

In the terminology of Section 1.2, this theorem implies that if the class of odd-hole-free graphs is χ-bounded, then its optimal χ-bounding function f satisfies $f(1) = f(2) = f(3) \leq 4$. (In fact, $\omega(G) = 1$ implies that G has no edge and thus $\chi(G) = 1$, and it is not hard to see that an odd-hole-free graph G with $\omega(G) = 2$ is bipartite and, thus, $\chi(G) = 2$. Hence, $f(1) = 1$, $f(2) = 2$ and $f(3) = 4$.)

Chudnovsky, Robertson, Seymour and Thomas proved Theorem 1.2.3 by first proving a structural description of K_4-free graphs with no odd hole. In Chapter 2, we use this structural description to investigate the presence of so-called *even pairs* in such graphs.

Definition. *In a graph G, an (unordered) pair of distinct vertices $\{u, v\}$ is said to be an* even pair *if every induced path between u and v in G has an even number of edges.*

Our interest in even pairs originates from the fact that they are useful in vertex coloring. In particular, let $\{u, v\}$ be an even pair in a graph G and construct the graph G/uv from G by deleting u and v, and adding a new vertex uv that is adjacent to precisely the vertices that are adjacent to at least one of u, v in G. We say that G/uv is constructed by *contracting* the even pair $\{u, v\}$. Fonlupt and Uhry **[27]** proved that contracting an even pair in a graph does not change its chromatic number. Even pairs are the basis for a few efficient vertex coloring algorithms. Moreover, Chudnovsky and Seymour **[15]** used even pairs to replace 55 pages of the (179-page) proof of the Strong Perfect Graph Theorem by a 9-page argument that is based on finding even pairs in perfect graphs.

A *clique cutset* in a connected graph G is a clique X in G such that $G \setminus X$ is disconnected. Clique cutsets are also useful in vertex coloring and this has the following reason. Suppose that G is a connected graph with a clique cutset X and suppose that we know how to color graphs with fewer vertices than G. Let $K_1, K_2, \ldots, K_p$ be the connected components of $G \setminus X$ and, for $i \in [p]$, let $G_i = G|(X \cup V(K_i))$. Since $|V(G_i)| < |V(G)|$, we may color each of the graphs G_i, $i \in [p]$. Perhaps by relabeling the colors, we may assume that the colorings have the property that each $x \in X$ receives the same color in each coloring. Now, we can combine these coloring to obtain a valid vertex coloring of G. Thus, $\chi(G) \leq \max\{\chi(G_1), \ldots, \chi(G_p)\}$.

Let $\bar{C}_7$ be the complement of a cycle of length seven, and let T_{11} be the graph with vertex set $\{v_1, v_2, \ldots, v_{11}\}$ such that for $i, j \in [11]$, v_i and v_j are adjacent if and only if $3 \leq |i - j| \leq 8$. We prove the following structure theorem for imperfect K_4-free graphs with no odd hole and no even pair:

Theorem 1.2.4. *Suppose that G is an imperfect connected K_4-free graph with no odd hole and no* *even pair. Then either G is isomorphic to one of $\{T_{11}, \bar{C}_7\}$, or G has a clique cutset.*

In the second part of Chapter 2, we investigate the circular chromatic number of such graphs:

Definition. *For a real number $r > 0$ and a graph G, a* circular r-coloring *of G is a function $c : V(G) \to [0, r)$ such that $1 \leq |c(u) - c(v)| \leq r - 1$ whenever $uv \in E(G)$. The* circular chromatic number *of G, denoted $\chi_c(G)$, is defined by $\chi_c(G) = \inf\{r : G$ has a circular r-coloring$\}$.*[3]

In plain English, the circular chromatic number of a graph is defined as follows. Consider putting the vertices of a graph G on a circle of circumference r, in such a way that whenever two distinct $u, v \in V(G)$ are adjacent, they are at at least distance one (on the circle) of each other. The circular chromatic number is defined as the smallest circumference r for which this is possible.

It is not hard to see that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ for every graph G. Thus, it follows from Theorem 1.2.3 that $\chi_c(G) \leq 4$ for every K_4-free graph with no odd hole. We use Theorem 1.2.4 and a linear programming duality argument to prove that, in fact, the circular chromatic number of a K_4-free graph with no odd hole is strictly less than 4:

Theorem 1.2.5. [62] *Let G be a K_4-free graph with no odd hole. Then $\chi_c(G) < 4$.*

We also construct an infinite class of graphs whose circular chromatic number is arbitrarily close to 4, demonstrating that the bound given in Theorem 1.2.5 is tight. Finally, one may ask the question whether K_5-free graphs with no odd hole have circular chromatic number that is strictly less that 5. We show that this is not true.

1.3 The Erdős-Hajnal conjecture

Another conjecture deals with large cliques or stable sets in classes of graphs that are characterized by excluding induced subgraphs. To be precise, we are interested in the *homogeneity number* $\hom(G)$ of a graph G, which is defined as

$$\hom(G) = \max\{\alpha(G), \omega(G)\}.$$

That is, the homogeneity number of a graph G is the size of a largest clique or stable set in G. A first result on the homogeneity number of a graph follows from the proof of Ramsey's theorem, Theorem 1.0.1. The proof establishes that $R(s, t) \leq \binom{s+t-2}{s-1}$, which implies the following corollary:

Corollary. $\hom(G) \geq \frac{1}{2} \log_2 |V(G)|$ *for every graph G.*

Paul Erdős proved that this bound is correct up to a multiplicative constant:

Theorem. [24] *For every $n \geq 1$, there exists a graph G with $|V(G)| = n$ such that* $\hom(G) \leq$

[3]Notice that in contrast to the usual (vertex) chromatic number, the circular chromatic is not necessarily an integer and in fact is in general a rational number **[59]**.

$2 \log |V(G)|$.

The proof of this theorem relies on the simplest conceivable type of a *random graph*: let $\mathcal{G}(n,p)$ be the class of random graphs G that are constructed as follows. The vertex set $V(G)$ satisfies $|V(G)| = n$. For every two distinct vertices $u, v \in V$, let uv be an edge with probability p. The events that two pairs of distinct vertices are edges are probabilistically independent of each other.

Erdős proved that with strictly positive probability, a random graph $G \in \mathcal{G}(n, \frac{1}{2})$ has no clique or stable set of size greater than $2 \log n$, thus proving that for every $n \in \mathbb{N}$ there exists a graph G on n vertices such that G has no clique or stable set has size greater than $2 \log n$. It is known, however, that such random graphs contain every fixed graph as an induced subgraph with high probability: (see *e.g.*, **[22]**)

Theorem. *Let $p \in (0,1)$, let H be a graph and, for each $n \geq 1$, let $G_n \in \mathcal{G}(n,p)$. Then*

$$\lim_{n\to\infty} \mathbb{P}(G_n \textit{ contains } H \textit{ as an induced subgraph}) = 1.$$

It is therefore natural to ask the following question: does excluding a fixed graph H as an induced subgraph guarantee the existence of a clique or a stable set that is larger than $\Omega(\log n)$? Erdős and Hajnal **[26]** proved that the answer is 'yes' by proving the following theorem:

Theorem. *For every graph H, there exists $c(H) > 0$ such that for every graph $G \in \mathrm{Forb}(H)$,*

$$\mathrm{hom}(G) \geq e^{c(H)\sqrt{\log |V(G)|}}.$$

However, they conjectured in the same paper that the correct lower bound is $\Omega(n^c)$ (*i.e.*, the factor $\sqrt{\log n}$ in 1.3 may be replaced by just $\log n$). To be precise, they conjectured the following:

Conjecture 1.3.1. (Erdős and Hajnal **[26]**) *For any fixed graph H, there exists $c(H) > 0$ such that for every graph $G \in \mathrm{Forb}(H)$,*

$$\mathrm{hom}(G) \geq |V(G)|^{c(H)}.$$

We say that a graph H *has the Erdős-Hajnal property* if there exists $c(H) > 0$ such that every graph $G \in \mathrm{Forb}(H)$ satisfies $\mathrm{hom}(G) \geq |V(G)|^{c(H)}$. The conjecture states that *every* graph has the Erdős-Hajnal property. Despite many efforts, the property has been established for a quite limited class of graphs only. It has been established **[3, 26]** for all graphs H with $|V(H)| \leq 4$, but there are graphs with as few as five vertices for which the property is still unresolved. Some progress on larger graphs was made by Alon, Pach and Solymosi **[3]**, who proved that if H_1 and H_2 have the Erdős-Hajnal property, then so does the graph that is constructed from H_1 and H_2 by the substitution operation. Moreover, Chudnovsky and Safra **[12]** proved that that the triangle with two disjoint pendant edges (the so-called 'bull') has the property. Among the graphs H on five vertices, this already leaves two open cases: the cycle of length five C_5, and the four-edge path P_4.

1.3.1 Contributions

One of the main ideas of the paper by Chudnovsky and Safra **[12]** is to fractionally cover a bull-free graph with a 'small' number of perfect graphs. This idea can be expressed in terms of a property that we call *narrowness*.

Definition. *Let $\beta \geq 1$. We say that a graph G is β-*narrow *if every function $g : V(G) \to \mathbb{R}^+$ with the property that $\sum_{v \in V(P)} g(v) \leq 1$ whenever P is a perfect induced subgraph of G satisfies*

$$\sum_{v \in V(G)} \left[g(v)\right]^{\beta} \leq 1.$$

We prove in Chapter 3 that if a graph G is β-narrow, then it satisfies $\hom(G) \geq |V(G)|^{1/2\beta}$. Thus, for a fixed graph H, if every graph in $\mathrm{Forb}(H)$ is β-narrow for some $\beta \geq 1$, then H has the Erdős-Hajnal property (with $c(H) = 1/2\beta$). In fact, the converse is also true. This implies that Conjecture 1.3.1 is equivalent to the following conjecture:

Conjecture. [16] *Let H be a graph. Then, there exists $\beta(H) \geq 1$ such that every $G \in \mathrm{Forb}(H)$ is $\beta(H)$-narrow.*

As said, one of the smallest graphs for which the Erdős-Hajnal property has not been established yet is the four-edge path P_4. A nice property of some graphs for which the Erdős-Hajnal property has been established is the fact that they are self-complementary (*e.g.*, the bull and the three-edge path P_3). This lead us to believe that instead of considering graphs in $\mathrm{Forb}(P_4)$ (which seems quite hard), it may be easier to start by excluding both the four-edge path and its complement. Indeed, in Section 3.3, we give a simple proof for the fact that every graph $G \in \mathrm{Forb}(P_4, P_4^c)$ is $(\log_4 5)$-narrow, using a theorem of Fouquet **[28]**. This implies that $\hom(G) \geq |V(G)|^{1/2\log_4 5} \geq |V(G)|^{0.43}$ for all $G \in \mathrm{Forb}(P_4, P_4^c)$. Unfortunately, proving this bound for $\mathrm{Forb}(P_4, P_4^c)$ does not seem to help much in establishing the Erdős-Hajnal property for P_4. The main result of Chapter 3 deals with the case when we exclude a four-edge path and the complement of a five-edge path. In particular, we prove the following theorem:

Theorem. (With Chudnovsky [16]) *Every graph $G \in \mathrm{Forb}(P_4, P_5^c)$ is 3-narrow and satisfies $\hom(G) \geq |V(G)|^{1/6}$.*

1.4 Forbidden induced subgraphs in engineering: fractional strong perfection

In addition to the previous results which are mostly interesting from a theoretical viewpoint, we present here some graph theoretical results that actually answer some questions about an electrical engineering problem.

We say that a graph G is *strongly perfect* if every induced subgraph H of G contains a stable set that meets every (inclusion-wise) maximal clique of H. Strongly perfect graphs were first studied by Berge and Duchet **[5]** as a special class of perfect graphs. They form a natural subclass of perfect graphs in the following sense: every perfect graph (and hence each of its induced subgraphs) contains a stable set that meets every maximum cardinality clique. Strongly perfect graphs satisfy the stronger property that they contain a stable set meeting every inclusion-wise maximal clique. In Chapters 4 and 5, we are interested in a fractional relaxation of strong perfection:

Definition. *A graph G is said to be* fractionally strongly perfect *if for every induced subgraph H of G there exists a vertex weighting $g : V(H) \to \mathbb{R}_+$ such that $\sum_{v \in K} g(v) = 1$ for every (inclusion-wise) maximal clique K in H.*

(It is not hard to see that by requiring the weights to be integral, we obtain a definition that is equivalent to strong perfection). Before we discuss our contributions on this topic, we will first give a motivation for studying this class of graphs. This motivation lies in the following application in wireless network communication in electrical engineering. Since we are actually interested in graphs of which the complement is fractionally strongly perfect, we say that a graph G is *fractionally co-strongly perfect* if G^c is fractionally strongly perfect:

Definition. *A graph G is said to be* fractionally co-strongly perfect *if for every induced subgraph H of G there exists a vertex weighting $g : V(H) \to \mathbb{R}_+$ such that $\sum_{v \in S} g(v) = 1$ for every (inclusion-wise) maximal stable set S in H.*

1.4.1 Motivation – wireless network communication

Consider a wireless communication network $H = (V, E)$, in which the vertices in set V represent agents (*i.e.*, transmitters and receivers), and $E \subseteq \{ij : i, j \in V, i \neq j\}$ is a set of connections representing pairs of agents between which data flow can occur. At each vertex of the network, information packets are received over time and these packets must be transmitted to their destination (*i.e.*, an adjacent vertex). We assume that time is slotted and that packets are of equal size, each packet requiring one time slot of service across a link. A queue is associated with each edge in the network, representing the packets waiting to be transmitted on this link.

An important issue in operating a wireless communication network is that two connections might interfere with each other. One way to model this is to define the so-called *interference graph G of H*, whose vertices are the edges of H and in which two vertices are adjacent if the corresponding edges in H are not allowed to send data simultaneously because of interference constraints. One of the simplest interference models that can be found in the literature is the so-called *primary interference model*. This model states that two connections interfere with each other if and only if the corresponding edges share a vertex in H. Therefore, under this interference model, the interference graph G is exactly the line graph of H. Although certainly interesting from a theoretical perspective, it is known that the primary interference model does not always reflect reality accurately. There is therefore a substantial literature dealing with more general interference models, many of which fit into the interference graph framework.

A *scheduling algorithm* selects a set of edges to activate at each time slot, and transmits packets on those edges. Since they must not interfere, the selected edges must form a stable set in the interference graph G. A scheduling algorithm is called *stable on* G if, informally speaking, the sizes of the queues do not grow to infinity when this algorithm is adopted (we refer the reader to Chapter 4 for a formal definition of stability).

It was shown in **[53]** that the Maximum Weight Stable Set algorithm (MWSS) that selects a stable set in the interference graph that corresponds to the connections in the network with the largest total queue sizes at each slot is stable for every interference graph G. Although this MWSS algorithm is stable, it is not a tractable algorithm in many situations because it needs centralized computing of an optimal solution. Hence, there has been an increasing interest in simple and potentially distributed algorithms. One example of such an algorithm is known as the Greedy Maximal Scheduling (GMS) algorithm **[34, 42]**. This algorithm greedily selects the set of served links according to the queue lengths at these links (*i.e.*, GMS greedily selects a maximal weight stable set in the interference graph). A drawback of using this algorithm is that, in general, it is not stable for all graphs for which MWSS is stable. However, **[23]** gave the following sufficient condition on interference graphs on which the GMS algorithm is stable:

Theorem. **[23]** *Let* G *be a fractionally co-strongly perfect graph. Then GMS is stable on* G*.*

Thus, by characterizing graphs that are fractionally co-strongly perfect, we characterize for which graphs GMS is stable.

1.4.2 Contributions

Our contributions consist of two parts. In Chapter 4, we obtain a structure theorem for line graphs that are fractionally co-strongly perfect. The graph theoretic tools used in Chapter 4 are relatively simple, and the chapter has more of an engineering flavor than the other chapters in this thesis. In addition to showing how fractional co-strong perfect shows up in the wireless networking context, its purpose is to serve as an introduction to the ideas used in Chapter 5, in which we present a generalization of the results of Chapter 4. This generalization is motivated by a (graph-theoretically natural) generalization of line graphs: the class of graphs that do not contain $K_{1,3}$ as an induced subgraph. Such graphs are call *claw-free*. In Chapter 5, we generalize the results of Chapter 4 to find a forbidden induced subgraph characterization of claw-free graphs that are fractionally co-strongly perfect.

Fractionally co-strongly perfect line graphs

In Chapter 4, we give a complete characterization of all line graphs that are fractionally co-strongly perfect (recall that line graphs form a natural class of graphs in wireless networking because they correspond to the simplest possible interference model). When considering a line graph G, it is natural to consider the graph H such that $G = L(H)$. In the setting, stable sets in G are equivalent to matchings in H and induced subgraphs in G are equivalent to subgraphs in H. Fractional strong perfection of G^c can be expressed as: for every subgraph H' of H, there exists a fractional weighting of the edges of H' such that every maximal matching in H' receives weight exactly one.

We obtain a forbidden subgraph characterization of all graphs H for which $L(H)$ is fractionally co-strongly perfect:

Theorem 1.4.1. (With Birand, Chudnovsky, Seymour, Ries, Zussman [6]) *Let H be a connected graph. $L(H)$ is fractionally co-strongly perfect if and only if H has no cycle of length six, or of length at least eight, and H has no two edge-disjoint cycles of length at least five as a subgraph.*

We prove Theorem 1.4.1 by means of a structure theorem that is stated in terms of the *block-decomposition* of the graph.

Definition. *Let H be a graph. We say that a maximal 2-connected subgraph of H is a* block *of H. The collection of blocks of H is called the* block-decomposition *of H.*

It is well-known (see *e.g.*, **[31]**) that the block-decomposition of a graph exists and is unique. For $t, n \geq 1$, let K_n denote a complete graph on n nodes and let $K_{n,t}$ denote an $n \times t$ complete bipartite graph. We denote by $K_{2,t}^+$ the graph that consists of a set X of two adjacent vertices and a stable set Y with $|Y| = t$ such that X is complete to Y. The structure theorem that we obtain for fractionally co-strongly complements line graphs reads as follows.

Theorem 1.4.2. (With Birand, Chudnovsky, Seymour, Ries, Zussman [6]) *Let H be a connected graph. Suppose that H has no cycle of length six, or of length at least eight, and H has no two edge-disjoint cycles of length at least five. Then, at most one block of H has a cycle of length five or seven, and all other blocks are isomorphic to K_2, K_3, K_4, or to $K_{2,t}$ or $K_{2,t}^+$ for some $t \geq 2$.*

Figure 4.2 depicts an example of a graph described in Theorem 1.4.2. In addition to this characterization, we give a linear-time recognition algorithm for such graphs.

Fractionally co-strongly perfect claw-free graphs

A natural generalization of line graphs is the class of claw-free graphs: a graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. Having described completely in Chapter 4 the structure of line graphs that are fractionally co-strongly perfect, it is natural to try to extend the characterization to claw-free graphs. This is what we do in Chapter 5.

In Chapter 5, we use a structure theorem for claw-free graphs of Chudnovsky and Seymour **[14]**, which states that every claw-free graph is either of a certain basic type, or it looks like a generalization of a line graph. The approach is similar to the one taken in Chapter 4, which we have included in this thesis for exactly that reason. It turns out that the forbidden induced subgraphs are precisely the cycle of length six, all cycles of length at least eight, four particular graphs, and a collection of graphs that are constructed by taking two graphs, each a copy of one of three particular graphs, and joining them by a path of arbitrary length in a certain way. Many of these forbidden induced subgraphs follow immediately from Theorem 1.4.1, but a new collection of forbidden induced subgraphs appears as well.

We denote by $\mathcal{F}$ the collection of these forbidden induced subgraphs (we refer the reader to beginning of Chapter 5 for the exact definition of $\mathcal{F}$), and we say that a graph G is $\mathcal{F}$-free if G contains none of the graphs in $\mathcal{F}$ as an induced subgraph. We prove the following theorem:

Theorem. (With Chudnovsky and Ries [18, 19]) *Let G be a claw-free graph. Then, G is fractionally co-strongly perfect if and only if G is $\mathcal{F}$-free.*

Let us think about fractional co-strong perfection a bit more. Recall that a fractionally co-strongly perfect graph has a vertex-weighting such that every maximal stable set receives weight exactly one. We call such a vertex-weighting *saturating*.

What are sufficient conditions for a graph G to have such a vertex-weighting? Suppose G has a clique K that meets every maximal stable set in G (compare to the definition of strong perfect, but in the complement). Under these circumstances, we call K a *dominant clique*. If G has a dominant clique K, then, by putting weight 1 on every vertex in that clique and weight 0 on all other vertices, we clearly construct a saturating vertex-weighting. So dominant cliques are 'good'. Next, suppose that every maximal stable set of G has the same size k, say. Then, we may put weight $1/k$ on every vertex, thereby trivially ensuring that every maximal stable set receives weight one. It turns out that these two situations essentially suffice in finding saturating vertex-weightings:

Theorem 1.4.3. (With Chudnovsky and Ries [18, 19]) *Let G be a connected $\mathcal{F}$-free claw-free graph. Then, either*

(1) *G has a dominant clique, or*

(2) *every maximal stable set has the same size $k \in \{2, 3\}$, or*

(3) *G has a vertex that is adjacent to all other vertices in G.*

Claw-free graphs that are strongly perfect in the complement

Wang **[57]** gave a forbidden induced subgraph characterization of claw-free graphs that are strongly perfect. Using the results of Chapter 5, we obtain a characterization of claw-free graphs that are strongly perfect in the complement.

Consider again Theorem 1.4.3. We prove in Chapter 5 that outcome (2) is only necessary when G is not perfect. Therefore, if we insist on G being perfect, we only have outcomes (1) and (3). It is not hard to see that, in that case, (3) becomes superfluous. Thus, the theorem becomes:

Theorem. *Every perfect $\mathcal{F}$-free claw-free graph has a dominant clique.*

Since every induced subgraph of perfect $\mathcal{F}$-free claw-free graph is also perfect $\mathcal{F}$-free and claw-free, this means that every perfect $\mathcal{F}$-free claw-free graph is strongly perfect. Therefore, as a corollary of our result, we obtain a characterization for claw-free graphs that are strongly perfect in the complement: (see Figure 1.1 for a pictorial definition of the graph $\mathcal{G}_4$ and the skipping rope of type $(3, 3)$)

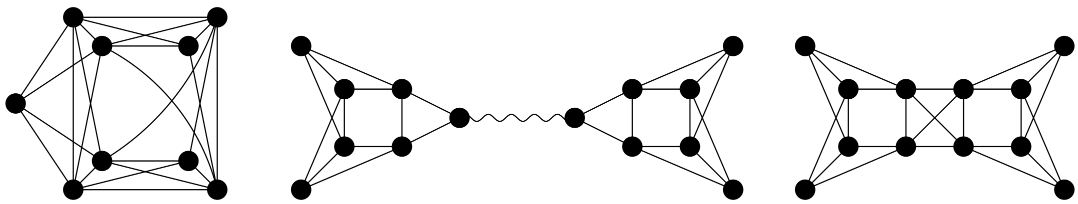


Figure 1.1: *The forbidden induced subgraphs for claw-free graphs with strongly perfect complements. Left: the graph $\mathcal{G}_4$. Center: the skipping rope of type $(3,3)$ of length $k \geq 1$ is obtained from this diagram by replacing the 'wiggly' edge by an induced k-edge path. Right: the skipping rope of type $(3,3)$ of length 0.*

Theorem. (With Chudnovsky and Ries [18, 19]) *Let G be a claw-free graph. G^c is strongly perfect if and only if G is perfect and no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even hole of length at least six, or a skipping rope of type $(3,3)$ of length $k \geq 0$.*

This theorem states that if a claw-free graph is perfect and it is fractionally co-strongly perfect, then it is integrally co-strongly perfect. We conjecture that this is true in general:

Conjecture 1.4.4. *If G is perfect and fractionally strongly perfect, then G is strongly perfect.*

2

K_4-free graphs with no odd hole: even pairs and the circular chromatic number

For an integer $n \geq 1$, let K_n denote the complete graph on n vertices. For $n \geq 1$, let $[n] = \{1, 2, \dots, n\}$. For two coprime integers p, q with $p \geq 2q$, $K_{p/q}$ is a graph with vertex set $\{v_1, v_2, \dots, v_p\}$ such that v_i and v_j are adjacent if and only if $q \leq |i - j| \leq p - q$ for $i, j \in [p]$. We call such a graph a *circular p/q-clique*. As special cases of circular cliques, define $\bar{C}_7 = K_{7/2}$ and $T_{11} = K_{11/3}$. An *odd hole* in a graph G is an induced cycle of odd length at least five in G.

For a positive real number r and a graph G, a *circular r-coloring of G* is a function $c : V(G) \to [0, r)$ such that $1 \leq |c(u) - c(v)| \leq r - 1$ whenever $uv \in E(G)$. The *circular chromatic number of G*, denoted $\chi_c(G)$, is defined by $\chi_c(G) = \inf\{r : G \text{ has a circular } r\text{-coloring}\}$. The circular chromatic number was introduced by A. Vince in **[55]** as a refinement of the usual (vertex) chromatic number of a graph. For clarity we note that, in contrast to the usual chromatic number, the circular chromatic is not necessarily an integer and in fact is in general a rational number **[59]**. It follows from the definition of $\chi_c(G)$ that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. We call a circular $\chi_c(G)$-coloring an *optimal circular coloring* of G. It was shown in **[55]** that an optimal circular coloring always exists. We refer to **[59, 60]** for good surveys on the circular chromatic number.

Main results and organization of this chapter

This chapter deals with K_4-free graphs with no odd holes, and it consists of two parts. In the first part (Section 2.1), we investigate even pairs in K_4-free graphs with no odd holes. Our interest in even pairs originates in the fact that they are useful in graph coloring: contracting (see Section 2.2) an even pair in a graph does not change its chromatic number **[27]**. We will use the characterization of K_4-free graphs with no odd holes that was given in **[20]** to prove the following theorem about the structure of K_4-free graphs with no odd hole and no even pair:

Theorem 2.0.5. *Suppose that G is an imperfect K_4-free graph with no odd hole and no even pair.*

Then either G is isomorphic to one of $\{T_{11}, \bar{C}_7\}$, or G has a clique cutset.

The second part of this chapter is inspired by the following theorem from **[20]**:

Theorem. *Let G be a K_4-free graph with no odd hole. Then $\chi(G) \leq 4$.*

From this and the fact that $\chi_c(G) \leq \chi(G)$ for every graph G, it follows that the $\chi_c(G) \leq 4$ for every K_4-free graph G with no odd hole. In Section 2.2, we will use the results in Section 2.1 in conjunction with a linear programming argument to prove that this inequality is in fact strict:

Theorem 2.0.6. *Let G be a K_4-free graph with no odd hole. Then $\chi_c(G) < 4$.*

In Subsection 2.2.3 we will construct an infinite family of graphs whose circular chromatic number is arbitrarily close to 4, demonstrating that the bound given in Theorem 2.0.6 is tight.

2.1 Even pairs in K$_4$-free graphs with no odd hole

In order to prove Theorem 2.0.5, we will use the following structural theorem which is an immediate consequence of **3.1** and **9.1** in **[20]**. The definitions of the harmonious cutset, graphs of T_{11} type and graphs of the two heptagram types will be postponed until they are needed (they can also be found in **[20]**).

Theorem 2.1.1. [20] *Let G be an imperfect K_4-free graph with no odd hole. Then either G has a harmonious cutset, or G is of T_{11} type, or G is of the first or second heptagram type.*

We will analyze the presence of even pairs in K_4-free graphs with no odd hole by looking at the different outcomes of Theorem 2.1.1.

We say that a vertex u *dominates* another vertex v if $N(v) \subseteq N(u)$. We will repeatedly use the following observation:

(2.1.2) *Let G be a graph and let $u, v \in V(G)$ be two nonadjacent vertices. If u dominates v, then $\{u, v\}$ is an even pair in G.*

Proof. We claim that every induced path from u to v in G has exactly two edges. For suppose that there exists a path Q from u to v in G of length other than two. Since u and v are nonadjacent, this implies that Q has at least three edges. Let u', v' be the neighbors of u, v, respectively on Q. Since Q is induced and has at least three edges, u' is nonadjacent to v and v' is nonadjacent to u. But this contradicts the fact that u dominates v. This proves (2.1.2). ■

Throughout this section we will use a slightly nonstandard definition of a partition: for a set X, a *partition of* X is a collection $\{X_i\}_{i=1}^k$ of subsets of X such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^k X_i = X$.

We stress that, in contrast to the usual definition of a partition, we do not require the sets of a partition to be nonempty (unless explicitly stated).

2.1.1 Harmonious cutsets and graphs of T_{11} type

We call a cutset X in G *harmonious* if X can be partitioned into disjoint nonempty stable sets $X_1, X_2, \dots, X_k$ and

(S1) for all $i, j \in [k]$, if P is an induced path in G with one end in X_i and the other end in X_j, then P is even if $i = j$ and odd otherwise, and

(S2) if $k \geq 3$, then $X_1, X_2, \dots, X_k$ are pairwise complete to each other.

We call a cutset X in G a *special odd cutset* if $|X| = 2$, the two vertices in X are nonadjacent and every induced path in G between them is odd. The following lemma almost proves Theorem 2.0.5 for graphs with harmonious cutsets. The only undesired outcome is the special odd cutset, which will be handled later, in Section 2.1.3.

(2.1.3) *Let G be a graph with a harmonious cutset. Then either G has an even pair, or G has a clique cutset, or G has a special odd cutset.*

Proof. Let $X, X_1, \dots, X_k$ be as in the definition of the harmonious cutset. If $|X_i| \geq 2$ for some $i \in [k]$, any pair $\{u, v\} \subseteq X_i$ is an even pair by (S1). We may therefore assume that $|X_i| = 1$ for all $i \in [k]$. We may assume that X is not a clique, because if it is, then X is a clique cutset of size k and the lemma holds. It follows from (S1) and (S2) that X consists of two nonadjacent vertices such that every induced path between them is odd, so that X is a special odd cutset. This proves (2.1.3). ■

We say that a graph G is *of T_{11} type* if $V(G)$ can be partitioned into stable sets W_1, W_2, ..., W_{11} such that W_i is complete to W_j if and only if $3 \leq |i - j| \leq 8$ and anticomplete otherwise.

(2.1.4) *Suppose that G is a graph of T_{11} type. Then either G has an even pair, or G is isomorphic to T_{11}.*

Proof. Let $W_1, W_2, \dots, W_{11}$ be as in the definition of a graph of T_{11} type. If $|W_i| = 1$ for all $i \in [11]$, then G is isomorphic to T_{11}. Therefore, from the symmetry, we may assume that $|W_1| \geq 2$. Let $u, v \in W_1$ be distinct. Since $N(u) = N(v)$, it follows that u dominates v and hence from (2.1.2) it follows that $\{u, v\}$ is an even pair. This proves (2.1.4). ■

This handles the first two outcomes of Theorem 2.1.1. The next section is devoted to handling graphs of the first and second heptagram type.

2.1.2 Graphs of the first and second heptagram type

In this section, we will show that almost all graphs of the first and second heptagram type have an even pair, with graphs isomorphic to $\bar{C}_7$ as the only exception.

Graphs of the first heptagram type

We say that a graph G is of the *first heptagram type* if there exist $t \geq 1$ and a partition of $V(G)$ into ten stable sets $W_1, W_2, \dots, W_7, Y_2, Y_4, Y_7$ where Y_4, Y_7 may be empty but the other sets are nonempty, such that, with index arithmetic modulo 7:

(A1) for $1 \leq i \leq 7$, W_i is complete to W_{i+2} and anticomplete to W_{i+3}

(A2) for $i \in \{3, 4, 6, 7\}$, W_i is complete to W_{i+1}, and for $i = 1, 2$, W_i, W_{i+1} are linked; and every vertex in W_2 is complete to one of W_1, W_3

(A3) for $i = 4, 7$, every vertex in Y_i is complete to $W_{i+3} \cup W_{i-3}$, has a neighbor in W_i, and has no neighbor in W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}

(A4) Y_2, Y_4, Y_7 are pairwise anticomplete

(A5) there is a nonempty subset $C \subseteq W_2$ such that C is complete to $W_1 \cup W_3$, and Y_2, C are linked, and Y_2 is anticomplete to $W_2 \setminus C$

(A6) there exist partitions $M_0, M_1, \dots, M_t$ of W_5 and $N_0, N_1, \dots, N_t$ of W_6 where M_0, N_0 may be empty but the other sets are nonempty, such that for $1 \leq i \leq t$, M_i is complete to N_i, M_i is anticomplete to $W_6 \setminus N_i$, $W_5 \setminus M_i$ is anticomplete to N_i, and M_0, N_0 are linked (and consequently W_5, W_6 are linked)

(A7) there is a partition $X_1, X_2, \dots, X_t$ of Y_2 where $X_1, X_2, \dots, X_t$ are all nonempty, such that for $1 \leq i \leq t$, X_i is complete to $M_i \cup N_i$, and anticomplete to each of $W_5 \setminus M_i$, $W_6 \setminus N_i$, W_7, W_1, W_3, W_4.

Graphs of the first heptagram type trivially have an even pair:

(2.1.5) *Let G be a graph of the first heptagram type. Then G has an even pair.*

Proof. Let $W_2, W_4, W_5, W_6, M_1, N_1, X_1$ be as in the definition of the first heptagram type. Let $u \in X_1$ and $v \in W_4$. It follows from (A1) and (A2) that v is complete to $W_2 \cup W_5 \cup W_6$. Moreover, it follows from (A3), (A4), (A6) and (A7) that $N(u) \subseteq M_1 \cup N_1 \cup W_2 \subseteq W_2 \cup W_5 \cup W_6 \subseteq N(v)$. Therefore, v dominates u and it follows from (2.1.2) that $\{u, v\}$ is an even pair in G. This proves (2.1.5). ■

Graphs of the second heptagram type

Before we define the second heptagram type, let us say that a triple (W_1, W_2, W_3) of disjoint stable sets in G is a *crescent* if the following properties hold:

(C1) if $v_i \in W_i$ for $i = 1, 2, 3$ and v_2 is adjacent to both v_1 and v_3, then v_1 is adjacent to v_3

(C2) if $v_i \in W_i$ for $i = 1, 2, 3$ and v_2 is nonadjacent to both v_1 and v_3, then v_1 is nonadjacent to v_3.

We say that a graph G is of the *second heptagram type* if $V(G)$ can be partitioned into fourteen stable sets $W_1, W_2, \dots, W_7$ and $Y_1, Y_2, \dots, Y_7$ where $W_1, W_2, \dots, W_7$ are nonempty but $Y_1, Y_2, \dots, Y_7$ may be empty and the following properties (where arithmetic is modulo 7) hold:

(B1) for $1 \leq i \leq 7$, W_i is anticomplete to W_{i+3}

(B2) for $2 \leq i \leq 7$, W_i is complete to W_{i+2}, and W_1, W_2, W_3 are pairwise linked

(B3) (W_1, W_2, W_3) is a crescent, and if W_1 is not complete to W_3 then $Y_2, Y_5, Y_6 = \emptyset$

(B4) for $i \in \{3, 4, 6, 7\}$, W_i is complete to W_{i+1}; W_5, W_6 are linked

(B5) for $1 \leq i \leq 7$, every vertex in Y_i is complete to $W_{i+3} \cup W_{i-3}$, has a neighbor in W_i, and has no neighbor in $W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}$

(B6) for $1 \leq i \leq 7$, every vertex in W_i with a neighbor in Y_i is complete to $W_{i+1} \cup W_{i-1}$

(B7) for $1 \leq i \leq 7$, Y_i is complete to Y_{i+1} and anticomplete to $Y_{i+2} \cup Y_{i+3}$

(B8) for $1 \leq i \leq 7$, at least one of Y_i, Y_{i+1}, Y_{i+2} is empty.

For distinct vertices $u, v, u', v' \in V(G)$, we say that the ordered pairs (u, v) and (u', v') are *friends* in G if u is adjacent to u', v is adjacent to v', u is nonadjacent to v' and v is nonadjacent to u'. For two disjoint sets $A, B \subseteq V(G)$, we say that A is *friendly with* B if there exists $u, v \in A$ and $u', v' \in B$ such that (u, v) and (u', v') are friends. Note that being friendly is a symmetric relationship so that we can speak of two sets A and B being *friendly with each other*.

We distinguish between graphs of the second heptagram type that have sets W_i and W_{i+1} that are friendly with each other and ones that do not have such sets. Note that from (B4) it follows that it suffices to look for such friendly sets for $i \in \{1, 2, 5\}$.

Graphs of the second heptagram type with friends

We will start with two claims about graphs of the second heptagram type that have a friendly pair in consecutive W_i's and then deduce that such graphs have an even pair.

(2.1.6) *Let (W_1, W_2, W_3) be a crescent, suppose that W_1, W_2, W_3 are pairwise linked and that $(u, v) \in W_1 \times W_1$ and $(u', v') \in W_2 \times W_2$ are friends. Then W_3 can be partitioned into sets $S \neq \emptyset$ and T such that $\{u, v, u', v'\}$ is complete to S and anticomplete to T.*

Proof. Let $x \in W_3$. Suppose that x is adjacent to one of $\{u, v\}$. From the symmetry, we may assume that x is adjacent to u. By (C2) applied to u, v', x, it follows that x is adjacent to v'. By (C1) applied to v, v', x, it follows that x is adjacent to v. So x is complete to $\{u, v\}$. Therefore, W_3 can be partitioned into sets S and T such that S is complete to $\{u, v\}$ and T is anticomplete to $\{u, v\}$. It follows from the fact that W_1 and W_3 are linked that $S \neq \emptyset$. Now let $s \in S$. By (C2) applied to u, v', s and to v, u', s, it follows that s is adjacent to u', v' and hence S is complete to $\{u, v, u', v'\}$. Next, let $t \in T$. By (C1) applied to u, u', t and to v, v', t, it follows that t is nonadjacent to u', v' and hence T is anticomplete to $\{u, v, u', v'\}$. This proves (2.1.6). ■

(2.1.7) *Let G be a graph of the second heptagram type. Let $i \in \{1, 2, 5\}$ and suppose that $(u, v) \in W_i \times W_i$ and $(u', v') \in W_{i+1} \times W_{i+1}$ are friends. Then $N(u)\backslash W_{i+1} = N(v)\backslash W_{i+1}$ and $N(u') \setminus W_i = N(v') \setminus W_i$.*

Proof. From the symmetry, we may assume that $i \in \{1, 5\}$. First suppose that $i = 5$. It suffices to show that $N(u)\backslash W_6 = N(v)\backslash W_6$. We first note that from (B1) and (B5), it follows that $N(u)\backslash W_6$ and $N(v)\backslash W_6$ are subsets of $W_3 \cup W_4 \cup W_7 \cup Y_1 \cup Y_2 \cup Y_5$. It follows from (B2) and (B5) that u and v are complete to $W_3 \cup W_4 \cup W_7 \cup Y_1 \cup Y_2$. It follows from (B6) and the fact that (u, v) and (u', v') are friends that $\{u, v\}$ is anticomplete to Y_5. Therefore, $N(u)\backslash W_6 = N(v)\backslash W_6 = W_3 \cup W_4 \cup W_7 \cup Y_1 \cup Y_2$. This proves the claim when $i = 5$.

So we may assume that $i = 1$. We first claim that $N(u)\backslash W_2 = N(v)\backslash W_2$. It follows from (B1) and (B5) that $N(u)\backslash W_2$ and $N(v)\backslash W_2$ are subsets of $W_3 \cup W_6 \cup W_7 \cup Y_1 \cup Y_4 \cup Y_5$. It follows from (B2) and (B5) that u and v are complete to $W_6 \cup W_7 \cup Y_4 \cup Y_5$. It follows from (B6) and the fact that (u, v) and (u', v') are friends that $\{u, v\}$ is anticomplete to Y_1. Finally, it follows from (2.1.6) that W_3 can be partitioned into sets S, T such that S is complete to $\{u, v\}$ and T is anticomplete to $\{u, v\}$. Therefore, $N(u)\backslash W_2 = N(v)\backslash W_2 = W_6 \cup W_7 \cup Y_4 \cup Y_5 \cup S$. Next, we claim that $N(u') \setminus W_1 = N(v')\backslash W_1$. It follows from (B1) and (B5) that $N(u')\backslash W_1$ and $N(v')\backslash W_1$ are subsets of $W_3 \cup W_4 \cup W_7 \cup Y_2 \cup Y_5 \cup Y_6$. It follows from (B2) and (B5) that u' and v' are complete to $W_4 \cup W_7 \cup Y_5 \cup Y_6$. It follows from (B6) and the fact that (u, v) and (u', v') are friends that $\{u, v\}$ is anticomplete to Y_2. Finally, it follows from (2.1.6) that W_3 can be partitioned into sets S, T such that S is complete to $\{u', v'\}$ and T is anticomplete to $\{u', v'\}$. In particular, $N(u') \cap W_3 = N(v') \cap W_3$. Therefore, $N(u)\backslash W_2 = N(v)\backslash W_2 = W_4 \cup W_7 \cup Y_5 \cup Y_6 \cup S$, thereby completing the proof of (2.1.7). ■

This enables us to find even pairs:

(2.1.8) *Let G be a graph of the second heptagram type. Suppose that W_i is friendly with W_{i+1} for some $i \in [7]$. Then G has an even pair.*

Proof. It follows from (B4) that $i \in \{1, 2, 5\}$. Let $(u, v) \in W_i \times W_i$ and $(u', v') \in W_{i+1} \times W_{i+1}$ be friends. For $q, q' \in [7]$, let $\mathcal{P}_{q,q'}$ be the set of odd induced paths with endpoints $a, b \in W_q$ such that there exist $a', b' \in W_{q'}$ such that (a, b) and (a', b') are friends. Let $\mathcal{P} = \mathcal{P}_{i,i+1} \cup \mathcal{P}_{i+1,i}$. We will show that $\mathcal{P} = \emptyset$, implying that there exists no odd induced path from u to v, and therefore that $\{u, v\}$ is an even pair.

Suppose for a contradiction that $\mathcal{P} \neq \emptyset$. Then there exists $P \in \mathcal{P}$ such that $|V(P)|$ is minimum. Let a and b be the endpoints of P. Let $\{j, k\} = \{i, i+1\}$ be such that $P \in \mathcal{P}_{j,k}$. It follows that $a, b \in W_j$. Let a' and b' be the neighbors of a and b, respectively, in P. It follows from the fact that P is an induced odd path that (a, b) and (a', b') are friends. From this and the fact that $N(a)\backslash W_k = N(b)\backslash W_k$ by (2.1.7), it follows that $a', b' \in W_k$. Now construct the path P' from P by deleting the endpoints a and b. Clearly, $P' \in \mathcal{P}$, but $|V(P')| < |V(P)|$, a contradiction. This proves that $\mathcal{P} = \emptyset$, thereby completing the proof of (2.1.8). ■

Graphs of the second heptagram type with no friends

We will now turn to graphs with no friends:

(2.1.9) *Let G be a graph of the second heptagram type. Let $i \in [7]$ and suppose that W_i is not friendly with W_{i+1}. Then there exists $u \in W_i$ that is complete to $W_{i+1} \cup W_{i+2}$.*

Proof. Let $u \in W_i$ be a vertex with a maximum number of neighbors in W_{i+1}. We first claim that u is complete to W_{i+1}. For suppose that there exists $v' \in W_{i+1}$ that is not adjacent to u. From (B2) and (B4), it follows that v' has a neighbor $v \in W_i$. By the choice of u, there exists $u' \in W_{i+1}$ that is adjacent to u but not to v. But now (u, v) is friends with (u', v'), contradicting the fact that W_i is not friendly with W_{i+1}. Therefore u is complete to W_{i+1}. If $i \neq 1$, it follows from (B2) that u is complete to W_{i+2}. If $i = 1$, let $x \in W_3$ be given. From (B2) it follows that x has a neighbor $v \in W_2$. From (C1) applied to u, v, x, it follows that u and x are adjacent. Therefore u is complete to W_{i+2}. This proves (2.1.9). ■

(2.1.10) *Let G be a graph of the second heptagram type. Suppose that for each $i \in [7]$, W_i is not friendly with W_{i+1}. Then either G has an even pair, or G is isomorphic to $\bar{C}_7$.*

Proof. We may assume that G is not isomorphic to $\bar{C}_7$.

(i) *Let $i \in [7]$. Then there exists $u \in W_i$ such that u is complete to $W_{i-2} \cup W_{i+1} \cup W_{i+2}$ and, if $i \neq 2$, then u is complete to $W_{i-2} \cup W_{i-1} \cup W_{i+1} \cup W_{i+2}$.*

From the symmetry, we may assume that $i \in \{1, 2, 4, 5\}$. It follows from (2.1.9) that there exists $u \in W_i$ such that u is complete to $W_{i+1} \cup W_{i+2}$. It follows from (B2) that W_i is complete to W_{i-2} (since $i \neq 3$). This proves the claim if $i = 2$. So we may assume that $i \neq 2$. It follows from (B4) that u is complete to W_{i-1}. This proves **(i)**. □

(ii) *If $Y_i \neq \emptyset$ for some $i \in [7]$, then G has an even pair.*

Choose any $y \in Y_i$. It follows from property (B8) and the symmetry that we may assume that $Y_{i-1} = \emptyset$. By **(i)**, we may choose $u \in W_{i-2}$ such that u is complete to $W_i \cup W_{i-3} \cup W_{i+3}$.

It follows from (B5) that u is complete to Y_{i+1}. Therefore, by (B5) and (B7), $N(y) \subseteq W_i \cup W_{i-3} \cup W_{i+3} \cup Y_{i+1} \subseteq N(u)$, and u and y are nonadjacent. Hence it follows from (2.1.2) that $\{u, y\}$ is an even pair. This proves **(ii)**. □

In the light of **(ii)**, we may now assume that $Y_i = \emptyset$ for all $i \in [7]$. If $|W_i| \geq 2$ for some $i \in \{1, 3, 4, \dots, 7\}$, then by **(i)** we may choose $u \in W_i$ such that u is complete to $W_{i-2} \cup W_{i-1} \cup W_{i+1} \cup W_{i+2}$ and choose any other $v \in W_i$. It follows that u dominates v and hence, by (2.1.2), that $\{u, v\}$ is an even pair. So we may assume that $|W_i| = 1$ for every $i \in \{1, 3, 4, \dots, 7\}$. Since G is not isomorphic to $\bar{C}_7$, it follows that $|W_2| \geq 2$. Now choose $u, v \in W_2$. It follows from the definition of the second heptagram type that $N(u) = N(v)$ and hence it follows from (2.1.2) that $\{u, v\}$ is an even pair. This proves (2.1.10). ■

2.1.3 Proof of Theorem 2.0.5

The lemmas from Sections 2.1.1 and 2.1.2 lead to the following structural result:

(2.1.11) *Suppose that G is an imperfect K_4-free graph with no odd hole. Then either*

1. *G is isomorphic to one of $\{T_{11}, \bar{C}_7\}$, or*
2. *G has an even pair, or*
3. *G has a clique cutset, or*
4. *G has a special odd cutset.*

Proof. It follows from Theorem 2.1.1 that either G has a harmonious cutset, or G is of T_{11} type, or G is of the first or the second heptagram type. If G has a harmonious cutset, is of T_{11} type, or is of the first heptagram type, then the result follows from (2.1.3), (2.1.4), (2.1.5), respectively. If G is of the second heptagram type and W_i friendly with W_{i+1} for some $i \in [7]$, then the result follows from (2.1.8). If G is of the second heptagram type and no such i exists, then the result follows from (2.1.10). This proves (2.1.11). ■

We finish the proof of Theorem 2.0.5 by showing that outcome 4 in (2.1.11) is redundant:

Proof of Theorem 2.0.5. Let G be an imperfect K_4-free graph with no odd hole with $|V(G)|$ minimum such that G has a special odd cutset $\{u, v\}$ and none of the outcomes 1, 2, 3 of (2.1.11) hold. Recall that, by the definition of a special odd cutset, u and v are nonadjacent in G and every path between u and v is odd. Let

$$\mathcal{V} = \left\{ (V_1, V_2) \;\middle|\; \begin{array}{c} V_1, V_2 \subsetneq V(G) \backslash \{u, v\}, \\ (V_1, V_2) \text{ is a partition of } V(G) \backslash \{u, v\}, \\ \text{and } V_1 \text{ is anticomplete to } V_2 \end{array} \right\}.$$

Since $\{u, v\}$ is a cutset, $\mathcal{V}$ is nonempty. For every partition $(V_1, V_2) \in \mathcal{V}$ and for $i \in \{1, 2\}$, let $G_i(V_1, V_2)$ be the graph constructed from $G|(V_i \cup \{u, v\})$ by adding an edge between u and v. Note that since $\{u, v\}$ is a special odd cutset, $G_1(V_1, V_2)$ and $G_2(V_1, V_2)$ are both K_4-free graphs with no odd hole. From $\mathcal{V}$ choose a partition (V_1, V_2) such that the graph $G_1^* = G_1(V_1, V_2)$ is imperfect and, subject to this, $|V_1|$ is minimum. Such a partition exists because it was shown in **[54]** that if G_1 and G_2 defined as above are both perfect, then G is also perfect, contrary to the assumption that G is imperfect. Since G_1^* is imperfect, it follows from (2.1.11) and the choice of G that either G_1^* is isomorphic to one of T_{11} and $\bar{C}_7$, or G_1^* has an even pair, or G_1^* has a clique cutset.

First suppose that G_1^* isomorphic to T_{11} or $\bar{C}_7$. Since in both T_{11} and $\bar{C}_7$ every two adjacent vertices have a common neighbor, u and v have a common neighbor $x \in V_1$. It follows that $\{u, x, v\}$ induces a two-edge path in G from u to v, contradicting the fact that every path from u to v is odd. So G_1^* is not isomorphic to T_{11} or to $\bar{C}_7$.

Next suppose that G_1^* has an even pair $\{a, b\}$. Let P be an induced path from a to b in G. We claim that P is even. If $\{u, v\} \nsubseteq V(P)$, then, because $a, b \in V(G_1^*)$, it follows that P is also an induced path in G_1^*, and hence that P is even. So we may assume that $\{u, v\} \subseteq V(P)$. From the symmetry in u, v, we may assume that there are induced paths P_1, P_2, P_3 in G such that $P = a\text{-}P_1\text{-}u\text{-}P_2\text{-}v\text{-}P_3\text{-}b$. Since $\{a, b\}$ is an even pair in G_1^* and $G|((V(P) \setminus V(P_2) \cup \{u, v\})$ is an induced path between a and b in G_1^*, it follows that $|E(P_1)| + |E(P_3)|$ is odd. Moreover, since $\{u, v\}$ is a special odd cutset and P_2 is an induced path between u and v in G, it follows that $|E(P_2)|$ is odd. Hence, $|E(P)| = |E(P_1)| + |E(P_2)| + |E(P_3)|$ is even. This proves that every induced path in G between a and b is even and, therefore, that $\{a, b\}$ is an even pair in G, contrary to our assumption that G does not have an even pair.

Finally assume that G_1^* has a clique cutset X. Let (C_1, C_2) with $C_1, C_2 \subsetneq V(G_1^*)\backslash X$ be a partition of $V(G_1^*)\backslash X$ such that C_1 is anticomplete to C_2 in G_1^*. If at most one of u and v is an element of X, then X is also a clique cutset in G, contrary to the assumption that G does not satisfy outcome 3 of (2.1.11). Therefore $\{u, v\} \subseteq X$. Since u and v do not have common neighbors (otherwise there exists a two-edge path in G between u and v) and X is a clique, it follows that $X = \{u, v\}$. Since G_1^* is imperfect, at least one of $G_1^*|(C_1 \cup \{u, v\})$ and $G_1^*|(C_2 \cup \{u, v\})$ is (as shown in **[54]**). By the symmetry we may assume that $G_1^*|(C_1 \cup \{u, v\})$ is imperfect. But now $(C_1, C_2 \cup V_2) \in \mathcal{V}$ and $|C_1| < |V_1|$, contradicting the minimality of V_1.

This proves Theorem 2.0.5. ■

2.2 Circular coloring

In this section we will use the outcomes of Theorem 2.0.5 to show that every K_4-free graph with no odd hole has circular chromatic number strictly less than 4. It was shown in **[55]** that for coprime integers p, q with $p \geq 2q$, $\chi_c(K_{p/q}) = p/q$ and hence in particular we have that $\chi_c(\bar{C}_7) = 7/2$ and $\chi_c(T_{11}) = 11/3$. This immediately handles the first outcome of Theorem 2.0.5.

For a graph G and two vertices $x, y \in V(G)$, let G/xy be the graph obtained by deleting x and y and adding a new vertex xy adjacent to precisely $N(x) \cup N(y)$. This operation is called *contracting the pair* $\{x, y\}$. As said in the introduction, contracting even pairs does not decrease the circular chromatic number of the graph. In fact, contracting nonadjacent vertices does not decrease the circular chromatic number:

(2.2.1) *Let G be a graph and let $x, y \in V(G)$ be nonadjacent. Then, $\chi_c(G) \leq \chi_c(G/xy)$.*

Proof. Let $r = \chi_c(G/xy)$ and let $c : V(G/xy) \to [0, r)$ be a circular r-coloring of G/xy. It is straightforward to verify that $c' : V(G) \to [0, r)$ defined by $c'(u) = c(u)$ for all $u \in V(G/xy) \setminus \{xy\}$ and $c'(x) = c'(y) = c(xy)$ is a circular r-coloring of G. It follows that $\chi_c(G) \leq r$. This proves (2.2.1). ■

(We note that the graph H_n defined in the last part of Section 2.2.3 shows that the inequality in this Lemma is really an inequality, as opposed to its usual chromatic number counterpart **[46]** in which equality holds. This follows from the fact that glueing two graphs on an edge can be viewed as twice contracting an even pair.)

The following two subsections will be devoted to handling clique cutsets. We will start with a result about optimal circular coloring of large circular cliques. This will allow us to prove a lemma on the circular chromatic number of graphs that are obtained by "glueing" two copies of $K_{(tk-1)/k}$ (where $t \geq 3$ and $k \geq 1$ are integers) on an appropriately chosen clique. (The glueing operation will be made precise.) The result is the basis for showing that glueing two graphs that have circular chromatic number strictly less than 4 does not increase the circular chromatic number beyond 4, as long as we glue on triangles and edges. This handles clique cutsets, the second outcome of Theorem 2.0.5.

Throughout this section, we will use the following equivalent definition of the circular chromatic number **[9]**. Let G_1 and G_2 be graphs. We say that a function $f : V(G_1) \to V(G_2)$ is a *homomorphism from G_1 to G_2* if $f(u)f(v) \in E(G_2)$ whenever $uv \in E(G_1)$. For finite graphs,

$$\chi_c(G) = \inf\{p/q : \text{there exists a homomorphism from } G \text{ to } K_{p/q}\}.$$

The following theorem was implicitly proved in **[9]**:

Theorem 2.2.2. **[9]** *For coprime integers p, q with $p \geq 2q$, a graph G is circular p/q-colorable if and only if there exists a homomorphism from G to $K_{p/q}$.*

2.2.1 χ-Critical circular cliques in optimal circular colorings

Let G be a graph. For coprime integers p, q with $p \geq 2q$, we call an induced subgraph H of G a *χ-critical circular clique* if H is isomorphic to $K_{p/q}$ and $\chi(G) - 1 < p/q < \chi(G)$. Note that, by definition, for any χ-critical circular clique H, $\chi_c(H)$ is a noninteger larger than 2. We will start with a lemma that states that if $\chi_c(G) < \chi(G)$ then every χ-critical circular clique in G is optimally circularly colored either "clockwise" or "counterclockwise". To make this precise, let us introduce some notation. For a real number $r > 2$ and real numbers a, b, let $[a, b]_r$ denote the closed interval from a

to b in the cyclic group $\mathbb{R}/r\mathbb{Z}$. That is, writing $a' = a \pmod r$, $b' = b \pmod r$,

$$\text{if } a \le b\text{: } [a,b]_r = \begin{cases} [a', b'] & \text{if } a' \le b' \\ [b', r) \cup [0, a'] & \text{if } a' > b' \end{cases}$$
$$\text{if } a > b\text{: } [a,b]_r = [0,r) \backslash [b,a]_r.$$

Let the open interval $(a,b)_r$ and half-open intervals $[a,b)_r$, $(a,b]_r$ be defined in the obvious analogous way.

For a circular coloring $c : V(G) \to [0,r)$ of a graph G and $s \in \mathbb{R}$, define $T_s c : V(G) \to [0,r)$ by $T_s c(v) = c(v) + s \pmod r$. We say that an induced subgraph of G with vertices $\{v_1, v_2, \dots, v_n\}$ is *circularly colored clockwise (with respect to the circular coloring c)* if there exists an $s \in \mathbb{R}$ such that $T_s c(v_1) \le T_s c(v_2) \le \cdots \le T_s c(v_n)$. We call the function $r - c$ the *reversion* of the circular r-coloring c. We say that an induced subgraph is *circularly colored counterclockwise (w.r.t. c)* if it is circularly colored clockwise w.r.t. $r - c$. Note that if c is a circular r-coloring of G, then $T_s c$ and $r - c$ are circular r-colorings of G as well.

(2.2.3) *Let p, q be coprime integers with $p \ge 2q$. Let G be a graph with $\chi_c(G) < \chi(G)$, let H be a χ-critical circular clique in G with $\chi_c(H) = p/q$ and vertex set $\{v_1, v_2, \dots, v_p\}$ such that $v_i v_j \in E(G)$ if and only if $q \le |i-j| \le p-q$. Let c be an optimal circular coloring of G. Then H is either circularly colored clockwise or circularly colored counterclockwise with respect to c.*

Proof. Let $r = \chi_c(G)$. Note that, by the existence of a χ-critical circular clique, $r > 2$. Observe that for every $j \in [p]$ it follows from the fact that v_j is adjacent to each of $\{v_{j+q}, v_{j+q+1}, \dots, v_{j+p-q}\}$ that

$$c(v_i) \in [c(v_j) + 1, c(v_j) - 1]_r \text{ for all } i \in \{v_{j+q}, \dots, v_{j+p-q}\}. \tag{2.1}$$

(i) *Let $j \in [p]$ and let k be an integer such that $1 \le |k| \le q-1$. Then $c(v_{j+k}) \in (c(v_j)-1, c(v_j)+1)_r$.*

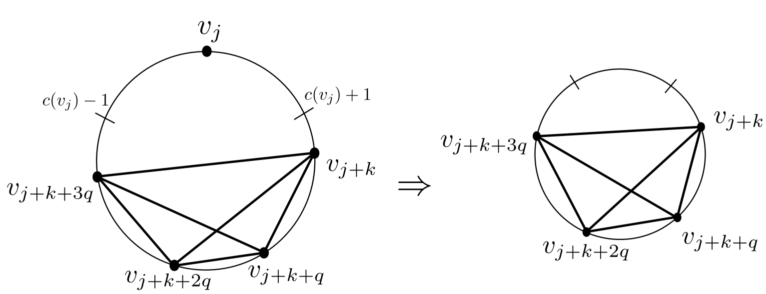


Figure 2.1: *Part* **(i)** *of (2.2.3). The diagram on the left shows the "colors" assigned to v_j, v_{j+k}, v_{j+k+q}, v_{j+k+2q}, v_{j+k+3q} on the circular interval $[0, r)_r$. The diagram on the right shows a circular $(r-1)$-coloring of $G|K$.*

From the symmetry, it suffices to show this for $1 \le k \le q-1$. Let $s = \lfloor p/q \rfloor$ and note that $r - 1 < \chi(G) - 1 \le \lfloor p/q \rfloor = s$. Now suppose that $c(v_{j+k}) \in [c(v_j)+1, c(v_j)-1]_r$. Since $(s-1)q < p-q$, the set $K = \{v_{j+k}, v_{j+k+q}, \dots, v_{j+k+(s-1)q}\}$ is a clique of size s in G (see Figure 2.1). From (2.1) and the assumption that $c(v_{j+k}) \in [c(v_j)+1, c(v_j)-1]_r$, it follows that K satisfies $c(u) \in [c(v_j)+1, c(v_j)-1]_r$ for all $u \in K$. But note that the length of the interval $[c(v_j)+1, c(v_j)-1]_r$ is $r-2$. Therefore we can construct from c a circular $(r-1)$-coloring of $G|K$ by replacing the interval $[c(v_j)-1, c(v_j)+1]_r$ by an interval of length 1 and restricting c to K. But since $r-1 < s$, this contradicts the fact that $\chi_c(G|K) = s$. This proves **(i)**. □

(ii) *Let $j \in [p]$. Then up to reversion of c, for every $k \in [q-1]$, $c(v_{j-k}) \in (c(v_j)-1, c(v_j)]_r$ and $c(v_{j+k}) \in [c(v_j), c(v_j)+1)_r$.*

By **(i)**, either $c(v_{j-q+1}) \in (c(v_j)-1, c(v_j)]_r$ or $c(v_{j-q+1}) \in [c(v_j), c(v_j)+1)_r$. By reversing c, we may assume that the former is the case. From **(i)** and the fact that v_{j-q+1} is adjacent to each of $v_{j+1}, \dots, v_{j+q-1}$, it follows that $c(v_{j+k}) \in [c(v_j), c(v_j)+1)_r$ for all $k \in [q-1]$. In turn, since in particular $c(v_{j+q-1}) \in [c(v_j), c(v_j)+1)_r$, it follows by the same reasoning that $c(v_{j-k}) \in (c(v_j)-1, c(v_j)]_r$ for all $k \in [q-1]$, proving **(ii)**. □

We can now prove the lemma. Possibly by taking the reversion of c and considering $T_{-c(v_1)}c$ instead of c, we may assume that $c(v_1) = 0$ and that $0 \le c(v_2) < 1$. We prove by induction on n that $c(v_1) \le c(v_2) \le \cdots \le c(v_n)$. Note that this is true for $n = 2$. Suppose it is true for $n = N$, $2 \le N \le p-1$. We claim that $c(v_N) \le c(v_{N+1})$. Since by **(ii)** $c(v_{N-1}) \in (c(v_N)-1, c(v_N)]_r$, it also follows from **(ii)** that $c(v_{N+1}) \in [c(v_N), c(v_N)+1)_r$. If $c(v_N) < r-1$, then we are done. So we may assume that $c(v_N) \ge r-1 > 1$. It remains to show that $c(v_{N+1}) \notin [0,1]_r$. For suppose that $c(v_{N+1}) \in [0,1]_r$. It follows from the fact that $c(v_1) = 0$ and (2.1) that $N+1 < q+1$ or $N+1 > 1+p-q$. If $N+1 < 1+q$, then $N < q$ and hence, by **(ii)** and the induction hypothesis, it follows that $0 \le c(v_N) < 1$, a contradiction. If $N+1 > 1+p-q$, then since $c(v_2) \in [0,1)$ and by **(ii)**, it follows that $c(v_{N+1}) \in (c(v_1)-1, c(v_1)]_r = (r-1, r]_r$, a contradiction. This completes the proof of (2.2.3). ■

2.2.2 Circular coloring and clique cutsets

Let H_1 and H_2 be two graphs with disjoint vertex sets, let T_1 and T_2 be cliques in H_1 and H_2, respectively, with $|T_1| = |T_2|$, and let $f : T_1 \to T_2$ be a bijective mapping. We define the *clique sum of* (H_1, T_1) *and* (H_2, T_2) *through* f as the graph G with vertex set $V(G) = (V(H_1) \cup V(H_2)) \setminus T_2$ and in which $u, v \in V(G)$ are adjacent if

1. $\{u, v\} \subseteq V(H_1)$ and $uv \in E(H_1)$; or
2. $\{u, v\} \subseteq V(H_2 \setminus T_2)$ and $uv \in E(H_2)$; or
3. $u \in T_1, v \in V(H_2 \setminus T_2)$ and $f(u)v \in E(H_2)$;

and nonadjacent otherwise. Note that our definition of the clique sum is nonstandard because we do not allow for deletion of clique edges. (However, this restriction is irrelevant for the analysis in this chapter since deleting edges from a graph does not increase its circular chromatic number.)

For motivation of this section, we note that, by definition, $\chi_c(G)$ is the optimal value of the following optimization problem with decision variables $r, \{x_v\}_{v\in V(G)}$:

$$\begin{aligned} \chi_c(G) = \min\ & r \\ \text{s.t. }\ & 1 \leq |x_u - x_v| \leq r-1, && \text{for all } uv \in E(G) \\ & 0 \leq x_v \leq r, && \text{for all } v \in V(G). \end{aligned} \tag{2.2}$$

There is an obvious one-to-one correspondence between the feasible points of this problem and the circular colorings of G. However, this problem is hard to deal with in general because of the $|x_u - x_v|$ term. Nevertheless, if the ordering of $\{x_v\}$ is known for some optimal solution, then each term $|x_u-x_v|$ can be replaced by x_u-x_v or x_v-x_u, depending on whether $x_u \geq x_v$ or $x_u \leq x_v$ in this optimal solution. Doing this turns the problem into a linear program, which is much easier to handle.

Suppose that G is a clique sum of two copies H_1 and H_2 of $K_{(tk-1)/k}$. Then from (2.2.3), it follows that if $\chi_c(G) < \chi(G)$, then for both H_1 and H_2 there are only two possible optimal circular colorings: clockwise and counterclockwise. From the symmetry, we may assume that H_1 is circularly colored clockwise and moreover that a (fixed) common vertex receives color 0. This means that we can recover the circular chromatic number by taking the minimum of the optimal values of two linear programs (corresponding to the cases where H_2 is circularly colored clockwise and circularly colored counterclockwise, respectively), as long as at least one of them has optimal value strictly less than $\chi(G)$. We use this idea to prove the following lemma. (In the proof of this lemma, we will in fact pick the correct one of the two linear programs.)

(2.2.4) *Let $t \geq 2$ and $k \geq 1$ be integers and let H_1, H_2 be two copies of $K_{(tk-1)/k}$ with disjoint vertex sets $\{u_1, u_2, \dots, u_{tk-1}\}$ and $\{v_1, v_2, \dots, v_{tk-1}\}$, respectively, such that $u_iu_j \in E(H_1) \iff v_iv_j \in E(H_2) \iff k \leq |i-j| \leq (t-1)k-1$. Let s, $a_1, a_2, \dots, a_s$, $b_1, b_2, \dots, b_s$ be integers such that $T_1 := \{u_{a_1}, u_{a_2}, \dots, u_{a_s}\}$ and $T_2 := \{v_{b_1}, v_{b_2}, \dots, v_{b_s}\}$ are cliques in H_1 and H_2, respectively, and assume that for every $j \in [s]$,*

$$(j-1)k < a_j \leq jk \quad \textit{and} \quad (j-1)k < b_j \leq jk. \tag{2.3}$$

Define the mapping $f : T_1 \to T_2$ by $f(u_{a_j}) = v_{b_j}$, for $j \in [s]$. Then the clique sum G of (H_1, T_1) and (H_2, T_2) through f satisfies $\chi_c(G) < t$.

Proof. For notational simplicity, let us identify u_{a_j} and v_{b_j} for each $j \in [s]$. From the symmetry, we may assume that $a_1 = b_1 = 1$. Also, let us define $n = tk - 1$. Consider the linear program $\mathcal{LP}_1$:

$$\begin{aligned} r^* = \min\ & r \\ \text{s.t. }\ & r + x_i - x_{i-k} \geq 1, && i \in [k] && [p_1, p_2, \dots, p_k] \\ & x_i - x_{i-k} \geq 1, && i \in \{k+1, k+2, \dots, n\} && [p_{k+1}, p_{k+2}, \dots, p_n] \end{aligned}$$

$$
\begin{array}{lll}
r + y_i - y_{i-k} \geq 1, & i \in [k] & [q_1, q_2, \dots, q_k] \\
y_i - y_{i-k} \geq 1, & i \in \{k+1, k+2, \dots, n\} & [q_{k+1}, q_{k+2}, \dots, q_n] \\
0 = x_1 \leq x_2 \leq \dots \leq x_n & & (*) \\
0 = y_1 \leq y_2 \leq \dots \leq y_n & & (*) \\
x_{a_i} = y_{b_i}, & i \in [s]. & [z_i]
\end{array}
$$

(the variables in square brackets will denote dual variables, see below.)

Let $\mathcal{LP}_2$ be the program $\mathcal{LP}_1$ but with the constraints marked with $(*)$ dropped. We claim that in order to prove the Lemma, it suffices to show that the optimal value of $\mathcal{LP}_2$ is strictly smaller than t. For let $(r^*, \mathbf{x}, \mathbf{y}) = (r^*, x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$ be an optimal solution of $\mathcal{LP}_2$ with $r^* < t$. It is easy to check that $(r^*, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = (r^*, 0, x_2 - x_1, \dots, x_n - x_1, 0, y_2 - y_1, \dots, y_n - y_1)$ is also an optimal solution of $\mathcal{LP}_2$. Moreover, it follows from the fact that $r^* < t$ and the first four sets of constraints that $(r^*, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ satisfies the inequality constraints marked with $(*)$. From this and the fact that the feasible region of $\mathcal{LP}_1$ is a subset of the feasible region of $\mathcal{LP}_2$, it follows that $(r^*, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is optimal for $\mathcal{LP}_1$. Now define the mapping $c : V(G) \to [0, r^*)$ by $c(u_i) = \tilde{x}_i$ and $c(v_i) = \tilde{y}_i$, $i \in [n]$. It is easy to check that c is a circular coloring of G and hence $\chi_c(G) \leq r^* < t$.

In order to show this, consider the linear programming dual problem of $\mathcal{LP}_2$, with decision variables $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n, z_1, z_2, \dots, z_s$:

$$
\begin{array}{rlll}
r^* = \max & \displaystyle\sum_{i=1}^{n} (p_i + q_i) & & \\
\text{s.t.} & p_i = p_{i+k}, & i \in [n] \backslash \{a_1, a_2, \dots, a_s\} & \text{(2.4a)} \\
& p_i = p_{i+k} - z_j, & i = a_j, j \in [s] & \text{(2.4b)} \\
& q_i = q_{i+k}, & i \in [n] \backslash \{b_1, b_2, \dots, b_s\} & \text{(2.4c)} \\
& q_i = q_{i+k} + z_j, & i = b_j, j \in [s] & \text{(2.4d)} \\
& \displaystyle\sum_{i=1}^{k} (p_i + q_i) = 1 & & \text{(2.4e)} \\
& p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n \geq 0 & &
\end{array}
$$

This dual can be interpreted as follows. Let us represent the vertices of H_1 by the following $t \times k$ "matrix" of vertices:

$$
M_u = \begin{bmatrix}
u_1 & u_2 & \cdots & u_{k-1} & u_k \\
u_{k+1} & u_{k+2} & \cdots & u_{2k-1} & u_{2k} \\
u_{2k+1} & u_{2k+2} & \cdots & u_{3k-1} & u_{3k} \\
\vdots & \vdots & & \vdots & \vdots \\
u_{(t-2)k+1} & u_{(t-2)k+2} & \cdots & u_{(t-1)k-1} & u_{(t-1)k} \\
u_{(t-1)k+1} & u_{(t-1)k+2} & \cdots & u_{tk-1} & \square
\end{bmatrix},
$$

where □ denotes an "empty" entry. For $j \in [s]$, let r_j^A and c_j^A denote the row and column index, respectively, of vertex u_{a_j} in this matrix. Since we asserted that $a_1 = 1$, we have $c_1^A = r_1^A = 1$. It follows from (2.3) that $r_j^A = j$ for each $j \in [s]$, and from the definition of $K_{(tk-1)/k}$, it follows that $c_j^A \geq c_{j-1}^A$ for $j \in \{2, \dots, s\}$.

For $j \in [s]$, consider a "walk" through the elements of M_u that starts at vertex u_{a_j} and moves down one row at a time, wrapping around to the next column and the first row whenever the bottom of the matrix is hit, until vertex $u_{a_{j+1}}$ is hit (where we let $a_{s+1} = 1$). Let A_j be the set of elements of M_u that are hit on this walk, except the starting vertex u_{a_j}, but including the final vertex $u_{a_{j+1}}$. Moreover, let $\bar{A}_j = A_j \cap \{u_1, u_2, \dots, u_k\}$. It is clear from the matrix representation that $A_1, A_2, \dots, A_s$ form a partition of $\{u_1, u_2, \dots, u_n\}$. Also, it is easy to see that

$$|\bar{A}_j| = c_{j+1}^A - c_j^A; \text{ and } |A_j| = |\bar{A}_j|t + 1. \tag{2.5}$$

For every $i \in [n]$, the value of p_i in a feasible solution of the dual problem can be thought of as a weight assigned to vertex u_i. Constraints (2.4a) and (2.4b) state that equal weight is assigned to each vertex in A_j. That is, for each $j \in [s]$, $p_i = w_j^A$ for all $u_i \in A_j$ for some $w_j^A \in \mathbb{R}$.

We define M_v, B_j and $\bar{B}_j$ analogously to M_u, A_j and $\bar{A}_j$, but replacing the roles of a, u, r_j^A, c_j^A by b, v, r_j^B, c_j^B, respectively. Constraints (2.4c) and (2.4d) state the analogues of (2.4a) and (2.4b) for B_j, q_i and w_j^B. Constraint (2.4e) states that the sum of the weights p_i and q_i assigned to the vertices of the first row of M_u and M_v equals 1.

With this interpretation in mind, the dual problem can be rewritten as the following linear program with decision variables $w_1^A, w_2^A, \dots, w_s^A, w_1^B, w_2^B, \dots, w_s^B, z_1, z_2, \dots, z_s$:

$$\begin{aligned} r^* = \max\ & \sum_{j=1}^{s} \big(|A_j| w_j^A + |B_j| w_j^B\big) & \\ \text{s.t.}\ & \sum_{j=1}^{s} \big(|\bar{A}_j| w_j^A + |\bar{B}_j| w_j^B\big) = 1 & \\ & w_j^A = w_{j+1}^A + z_j, & j \in [s] \\ & w_j^B = w_{j+1}^B - z_j, & j \in [s] \\ & w_j^A, w_j^B \geq 0, & j \in [s]. \end{aligned}$$

Using the facts that $\sum_{i=1}^s |A_j| = \sum_{i=1}^s |B_j| = n$ and $\sum_{i=1}^s |\bar{A}_j| = \sum_{i=1}^s |\bar{B}_j| = k$ and substituting the equality constraints into the objective function, this can be written as

$$\begin{aligned} r^* = \max\ & n(w_1^A + w_1^B) + \sum_{j=2}^{s} \big(|A_j| - |B_j|\big) z_j \\ \text{s.t.}\ & k(w_1^A + w_1^B) + \sum_{j=2}^{s} \big(|\bar{A}_j| - |\bar{B}_j|\big) z_j = 1 \end{aligned} \tag{2.6}$$

$$w_1^A, w_1^B \geq 0, \quad -w_1^A \leq \sum_{l=1}^{j} z_l \leq w_1^B, \quad j \in [s].$$

It follows from (2.5) and its counterpart for H_2 that for $j \in [s]$, $|A_j| - |B_j| = t(|\bar{A}_j| - |\bar{B}_j|)$. Using this and the fact that $n = tk - 1$, the objective function can be written as

$$tk(w_1^A + w_1^B) + t\sum_{j=2}^{s}(|\bar{A}_j| - |\bar{B}_j|)z_j - (w_1^A + w_1^B) \quad = \quad t - (w_1^A + w_1^B)$$

Dropping the subscripts from w_1^A and w_1^B, the problem becomes

$$\begin{aligned} r^* = \max\ & t - (w^A + w^B) \\ \text{s.t. } & k(w^A + w^B) + \sum_{j=2}^{s}(|\bar{A}_j| - |\bar{B}_j|)z_j = 1 \\ & w^A, w^B \geq 0, \quad -w^A \leq \sum_{l=1}^{j} z_l \leq w^B, \quad j \in [s]. \end{aligned}$$

It follows from linear programming duality that this problem is feasible (the primal problem has an obvious feasible solution). Clearly, since $w^A, w^B \geq 0$ for every feasible solution, we have $r^* \leq t$. Now suppose that there is a solution $(w^A, w^B, z_1, z_2, \dots, z_s)$ that has objective value exactly t. This solution satisfies $w^A + w^B = 0$ and hence, by the inequality constraints $w^A \geq 0, w^B \geq 0$, it follows that $w^A = w^B = 0$. Consequently, it follows that $z_j = 0$ for all $j \in [s]$. This however contradicts the first constraint. Hence there exists no feasible solution that has objective value greater or equal to t and therefore $r^* < t$, which is what had to be proved. This completes the proof of (2.2.4). ■

We need the following technical lemma.

(2.2.5) *Let $k \geq 1$ be an integer and let $s \in \{1, 2, 3\}$. Let G be a graph with $\chi_c(G) \leq 4 - 1/k$ and let $\{v_1, v_2, \dots, v_{4k-1}\}$ denote the vertices of $K_{(4k-1)/k}$ such that $v_i v_j \in E(K_{(4k-1)/k}) \iff k \leq |i - j| \leq 3k - 1$. Suppose that $\{t_1, t_2, \dots, t_s\}$ is a clique in G. Then there exists a homomorphism g from G to $K_{(4k-1)/k}$ and integers $a_1, \dots, a_s$ such that $g(t_j) = v_{a_j}$ and $(j-1)k < a_j \leq jk$, for every $j \in [s]$.*

Proof. By Theorem 2.2.2 and the choice of k, there exists a homomorphism g from G to $K_{(4k-1)/k}$. From the definition of a homomorphism, it follows that $g(\{t_1, \dots, t_s\})$ is a clique in $K_{(4k-1)/k}$. From the symmetry, we may assume that $g(t_1) = v_1$. Let $a_1 = 1$. If $s = 1$, the lemma holds. Otherwise, let a_2 be such that $g(t_2) = v_{a_2}$. From the symmetry we may assume in addition that $a_2 \leq 2k$. It follows that $a_2 > k$. If $s = 2$, the lemma holds. Let a_3 be such that $g(t_3) = a_3$. If $s = 3$, since $t_2 t_3$ and $t_1 t_3$ are edges, it follows that $2k < a_3 \leq 3k$. This completes the proof of (2.2.5). ■

We can now prove the main result of Subsection 2.2.2.

(2.2.6) *Let $s \in \{1, 2, 3\}$ and let H_1, H_2 be two K_4-free graphs. Let $T_1 \subseteq V(H_1)$ and $T_2 \subseteq V(H_2)$ be two cliques of size s, and let $f : T_1 \to T_2$ be a bijective mapping. Let G be the clique sum of (H_1, T_1) and (H_2, T_2) through f. If $\max\{\chi_c(H_1), \chi_c(H_2)\} < 4$, then $\chi_c(G) < 4$.*

Proof. Let k be an integer such that $\max\{\chi_c(H_1), \chi_c(H_2)\} \leq 4 - 1/k$ and let K_1 and K_2 be two disjoint a copies of $K_{(4k-1)/k}$ with vertex sets $\{u_1, u_2, \ldots, u_{4k-1}\}$ and $\{v_1, v_2, \ldots, v_{4k-1}\}$, respectively. Write $T_1 = \{r_1, r_2, \ldots, r_s\}$ and $T_2 = \{t_1, t_2, \ldots, t_s\}$ and assume without loss of generality that $f(r_j) = t_j$ for $j \in [s]$. From (2.2.5) it follows that there exist homomorphisms g_1, g_2 from H_1, H_2 to K_1, K_2, respectively, such that $g_1(r_j) = u_{a_j}$ and $g_2(t_j) = v_{b_j}$ satisfying $(j-1)k < a_j \leq jk$ and $(j-1)k < b_j \leq jk$ for $j \in [s]$.

Now consider the clique sum M of $(K_1, g(T_1))$ and $(K_2, g(T_2))$ through $g_2 \circ f \circ g_1^{-1}$. It follows from (2.2.4) that $\chi_c(M) < 4$ and hence there exists a homomorphism h from M to $K_{(4k'-1)/k'}$ for some integer $k' \geq 1$. Define the function $g : V(G) \to V(M)$ by $g(x) = g_1(x)$ if $x \in V(H_1)$ and $g(x) = g_2(x)$ if $x \in V(H_2) \backslash T_2$. It is easy to see that this is a homomorphism from G to M. Now $h \circ g$ is a homomorphism from G to $K_{(4k'-1)/k'}$ and hence $\chi_c(G) \leq \frac{4k'-1}{k'} < 4$ by Theorem 2.2.2. This proves (2.2.6). ■

2.2.3 Circular coloring of K_4-free graphs with no odd hole

We are now in a position to prove our second main result:

Proof of Theorem 2.0.6. We prove this by induction on $|V(G)|$. Let G be a K_4-free graph with no odd hole. If G is perfect, then, by definition of a perfect graph, $\chi_c(G) \leq \chi(G) = \omega(G) \leq 3$ and hence the theorem holds. Therefore we may assume that G is not perfect. By Theorem 2.0.5, either G is isomorphic to T_{11} or $\bar{C}_7$, or G has an even pair, or G has a clique cutset. If G is isomorphic to T_{11} or $\bar{C}_7$, then $\chi_c(G) \in \{11/3, 7/2\}$, and hence theorem holds. If G contains an even pair $\{x, y\}$, then consider G/xy. It is easy to see that G/xy is K_4-free and has no odd hole. Hence from (2.2.1) and the induction hypothesis it follows that $\chi_c(G) \leq \chi_c(G/xy) < 4$. If G has a clique cutset X, then let C_1 be a connected component of $G \backslash X$, let $C_2 = V(G) \backslash C_1$, let $H_1 = G|(C_1 \cup X)$ and let $H_2 = G|C_2$. Clearly, H_1 and H_2 are K_4-free and do not have odd holes and therefore it follows from the induction hypothesis that $\chi_c(H_i) < 4$ for $i = 1, 2$. Since G is the clique sum of (H_1, X) and (H_2, X) through the identity function $f : X \to X$, it follows from (2.2.6) that $\chi_c(G) < 4$. This proves Theorem 2.0.6. ■

We note that the requirement of forbidding odd holes is necessary, because if G is a triangle-free graph with $\chi(G) = 3$, then $\chi_c(M(G)) = 4$, where $M(G)$ denotes the Mycielskian of G (see **[11]**). Also, it is tempting to extrapolate the statement of this theorem and conjecture that, for every integer $p \geq 4$, if G is a K_p-free graph with no odd hole, then $\chi_c(G) < p$. However, this is already false for $p = 5$:

(2.2.7) *Let G_1, G_2 be two copies of $K_{14/3}$ with vertex sets $\{u_1, u_2, \ldots, u_{14}\}$, $\{v_1, v_2, \ldots, v_{14}\}$, respectively, such that $u_iu_j \in E(G_1) \iff v_iv_j \in E(G_2) \iff 3 \leq |i - j| \leq 11$. Define*

$f : \{u_1, u_4\} \to \{v_1, v_7\}$ by $f(u_1) = v_1$ and $f(u_4) = v_7$. Then the clique sum G of G_1 and G_2 through f satisfies $\chi_c(G) = 5$.

Proof. Since $\chi_c(G_1) = \chi_c(G_2) = 14/3$, it follows that $\chi(G_1) = \chi(G_2) = 5$. By vertex-coloring G_1 and G_2 separately and permuting the colors for G_2 so that the colors assigned to u_1 and v_1 and to u_4 and v_7 match, we can construct a 5-coloring of G by combining the colorings. It follows that $\chi_c(G) \leq \chi(G) \leq 5$. Now suppose that G has a circular r-coloring c with $r < 5$. We may assume that $c(v_1) = c(u_1) = 0$. From (2.2.3) and the symmetry, we may assume that $\{v_1, v_2, \dots, v_{14}\}$ is circularly colored clockwise and $\{u_1, u_2, \dots, u_{14}\}$ is either circularly colored clockwise or circularly colored counterclockwise. If $\{u_1, u_2, \dots, u_{14}\}$ is circularly colored clockwise, since u_1u_4 is an edge, it follows that $c(u_4) \geq 1$ and hence, since u_4u_7 is an edge, that $c(v_4) = c(u_7) \geq 2$. If $\{u_1, u_2, \dots, u_{14}\}$ is circularly colored counterclockwise, the same conclusion holds because u_1u_{11} and $u_{11}u_7$ are edges. But now from the fact that v_4v_7 and v_7v_{10} are edges and because $\{v_1, v_2, \dots, v_{14}\}$ is circularly colored clockwise, it follows that $c(v_{10}) \in [4, r)$. From the fact that $c(v_1) = 0$ and v_1v_{10} is an edge, it follows that $c(v_{10}) \in [1, r-1)$. But this is impossible because $r < 5$. This proves (2.2.7). ■

Finally, the bound in Theorem 2.0.6 is tight in the sense that there exists a sequence $\{H_n\}$ of K_4-free graphs with no odd hole such that $\chi_c(H_n) \to 4$ as $n \to \infty$. This sequence can be constructed as follows. Let H_1 be a copy of $\bar{C}_7$ with vertex set $\{v_1^1, v_2^1, \dots, v_7^1\}$. For $k \geq 2$, let G_k be a copy of $\bar{C}_7$ with vertex set $\{v_1^k, v_2^k, \dots, v_7^k\}$, let $T_1 = \{v_2^{k-1}, v_7^{k-1}\}$, let $T_2 = \{v_6^k, v_3^k\}$, and let $f : T_1 \to T_2$ be defined by $f(v_2^{k-1}) = v_6^k$ and $f(v_7^{k-1}) = v_3^k$. We define H_k as the clique sum of (H_{k-1}, T_1) and (G_k, T_2) through f. (See Figure 2.2) We have the following result.

Theorem 2.2.8. *For every $n \geq 1$, $\chi_c(H_n) \geq 4 - \frac{1}{n+1}$.*

Proof. Let $n \geq 1$ be given. Let $c : V(H_n) \to [0, r)$ be an optimal circular coloring of H_n. It follows from (2.2.6) that $r = 4 - \varepsilon$ for some $\varepsilon > 0$. From the symmetry, we may assume that $c(v_1^1) = 0$ and that $c(v_2^1) < r/2$. From this and from (2.2.3), it follows that $v_1^1, v_2^1, \dots, v_7^1$ are circularly colored clockwise. Let us first prove the following:

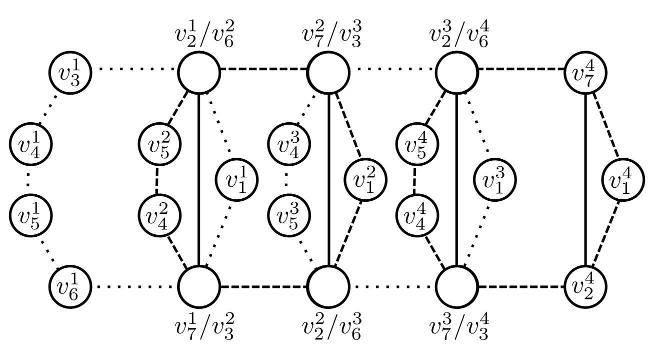


Figure 2.2: *The graph H_4 from the sequence $\{H_n\}_{n=1}^{\infty}$ that shows that the bound in Theorem 2.0.6 is tight. In this figure, every heptagon represents a copy of $\bar{C}_7$ and the dotted lines represents nonedges.*

$(*)$ *Let $1 \leq k \leq n$. If $k = 2p+1$ for some integer $p \geq 0$, then*

$$(1+p)\varepsilon \leq c(v_2^k) \leq 1-(1+p)\varepsilon \quad \text{and} \quad 3+p\varepsilon \leq c(v_7^k) \leq 4-(2+p)\varepsilon.$$

If $k = 2p$ for some integer $p \geq 1$, then

$$2+p\varepsilon \leq c(v_2^k) \leq 3-(1+p)\varepsilon \quad \text{and} \quad 1+p\varepsilon \leq c(v_7^k) \leq 2-(1+p)\varepsilon.$$

We will prove this by induction on k. First suppose that $k = 1$. Recall that $v_1^1, v_2^1, \dots, v_7^1$ are circularly colored clockwise. Hence, since v_1^1-v_3^1-v_5^1-v_7^1-v_2^1 and v_1^1-v_6^1-v_4^1-v_2^1-v_7^1 are paths, it follows that $3 \leq c(v_7^1) \leq 4-2\varepsilon$ and $\varepsilon \leq c(v_2^1) \leq 1-\varepsilon$. This proves **$(*)$** for $k = 1$.

Next suppose that $k = 2p+1$ for some integer $p \geq 1$. It follows from the induction hypothesis that $c(v_3^k) = c(v_7^{k-1}) \geq 1+p\varepsilon$ and $c(v_6^k) = c(v_2^{k-1}) \leq 3-(1+p)\varepsilon$. We first claim that $v_1^k, v_2^k, \dots, v_7^k$ are circularly colored clockwise. For suppose otherwise. Then by (2.2.3), $v_1^k, v_2^k, \dots, v_7^k$ are circularly colored counterclockwise. This, together with the fact that v_3^k-v_1^k-v_6^k is a path, implies that $c(v_6^k) \geq 3+p\varepsilon$, contrary to the fact that $c(v_6^k) \leq 3-(1+p)\varepsilon$. So $v_1^k, v_2^k, \dots, v_7^k$ are circularly colored clockwise. Hence, since v_3^k-v_5^k-v_7^k-v_2^k and v_6^k-v_4^k-v_2^k-v_7^k are paths, it follows that $3+p\varepsilon \leq c(v_7^k) \leq 4-(2+p)\varepsilon$ and $(1+p)\varepsilon \leq c(v_2^k) \leq 1-(1+p)\varepsilon$. This proves **$(*)$** for odd k.

Finally suppose that $k = 2p$ for some integer $p \geq 1$. It follows from the induction hypothesis that $c(v_3^k) = c(v_7^{k-1}) \geq 3+(p-1)\varepsilon$ and $c(v_6^k) = c(v_2^{k-1}) \leq 1-p\varepsilon$. We first claim that $v_1^k, v_2^k, \dots, v_7^k$ are circularly colored clockwise. For suppose otherwise. Then by (2.2.3), $v_1^k, v_2^k, \dots, v_7^k$ are circularly colored counterclockwise. This, together with the fact that v_3^k-v_1^k-v_6^k is a path, implies that $c(v_6^k) \geq 1+p\varepsilon$, contrary to the earlier observation that $c(v_6^k) \leq 1-p\varepsilon$. So $v_1^k, v_2^k, \dots, v_7^k$ are circularly colored clockwise. Hence, since v_3^k-v_5^k-v_7^k-v_2^k and v_6^k-v_4^k-v_2^k-v_7^k are paths, it follows that $1+p\varepsilon \leq c(v_7^k) \leq 2-(1+p)\varepsilon$. and $2+p\varepsilon \leq c(v_2^k) \leq 3-(1+p)\varepsilon$. This completes the proof of **$(*)$**.

We can now prove the theorem. If $n = 2p+1$ for some integer $p \geq 0$, it follows from **$(*)$** that $(1+p)\varepsilon \leq 1-(1+p)\varepsilon$, which is equivalent to $\varepsilon \leq \frac{1}{2(p+1)} = \frac{1}{n+1}$. If $n = 2p$ for some integer $p \geq 1$, then it follows from **$(*)$** that $2+p\varepsilon \leq 3-(1+p)\varepsilon$, which is equivalent to $\varepsilon \leq \frac{1}{1+2p} = \frac{1}{n+1}$. Finally, $\varepsilon \leq \frac{1}{n+1}$ implies that $\chi_c(H_n) \geq 4-\frac{1}{n+1}$, completing the proof of Theorem 2.2.8. ■

Note that using the techniques of (2.2.4), it can be shown that in fact $\chi_c(H_n) = 4-1/(n+1)$.

3

Large cliques or stable sets in graphs with no four-edge path and no five-edge path in the complement

For graphs $H_1, H_2, \dots, H_k$, let $\mathrm{Forb}(H_1, H_2, \dots, H_k)$ be the set of all graphs G such that for all $i \in [k]$, no induced subgraph of G is isomorphic to H_i. In **[26]**, Erdős and Hajnal made the following conjecture:

Conjecture 3.0.9. *For every graph H, there exists $\varepsilon(H) > 0$ such that every graph in* $\mathrm{Forb}(H)$ *has a clique or stable set of size at least $|V(G)|^{\varepsilon(H)}$.*

We say that a graph H has the *Erdős-Hajnal property* if there exists $\varepsilon(H) > 0$ such that every graph on n vertices that does not have H as an induced subgraph contains either a clique or a stable set of size at least $n^{\varepsilon(H)}$. Thus, Conjecture 3.0.9 is equivalent to asserting that all graphs have Erdős-Hajnal property.

Only a small number of graphs are currently known to have the Erdős-Hajnal property. Clearly, if H has the property, then so does its complement H^c. The most general known result from **[3]**, that states that if two graphs H_1 and H_2 have the Erdős-Hajnal property, then so does the graph constructed from H_1 by replacing a vertex $x \in V(H_1)$ by H_2 and making $V(H_2)$ complete to the neighbors of x in H_1 and anticomplete to the nonneighbors of x in H_1 (this operation is known as the substitution operation). On top of that, there are a few results on particular graphs. For example, all graphs on at most 4 vertices are known to have to property. Moreover, it was shown in **[12]** that the triangle with two disjoint pendant edges (this graph is known as the *bull*) has the property.

This leaves the four-edge-path P_4 and the cycle C_5 of length five as the remaining open cases for graphs on at most 5 vertices. This chapter deals with the case where H is a four-edge path, where, in addition, we exclude the complement of a five-edge path. To be precise, we will prove the following theorem:

Theorem 3.0.10. *Every graph $G \in \mathrm{Forb}(P_4, P_5^c) \cup \mathrm{Forb}(P_4^c, P_5)$ contains a clique or a stable set of size at least $|V(G)|^{1/6}$.*

Our approach will be similar to the one taken in **[12]** in which Chudnovsky and Safra prove that the bull has the Erdős-Hajnal property. In **[12]**, it was shown that every bull-free graph is 'narrow'. In this chapter, we generalize the concept of 'narrowness'. To be precise, we say that a function $g : V(G) \to \mathbb{R}^+$ is a *covering function for G* if $\sum_{p \in V(P)} g(p) \leq 1$ for every perfect induced subgraph P of G. For $\beta > 0$, we say that a graph G is β-narrow if $\sum_{v \in V(G)} \left[g(v)\right]^{\beta} \leq 1$ for every covering function g. Using this terminology, it was shown in **[12]** that bull-free graphs are 2-narrow. With this more general concept, we may define a new graph parameter which we call the *narrowness* of a graph:

Definition. *The* narrowness *of a graph G is denoted and defined by*

$$\nu(G) = \inf\{\beta : G \text{ is } \beta\text{-narrow}\}.$$

It is easy to see that $\nu(G) \geq 1$ for every (nonnull) graph G and $\nu(P) = 1$ for every perfect graph P. We will prove in Section 3.1 that every graph on n vertices is $(\log n)$-narrow, and thus $\nu(G)$ is finite for every graph G. Notice also that since a graph is perfect if and only if its complement is perfect, it follows that $\nu(G) = \nu(G^c)$ for every graph G. The narrowness of a graph is useful because of the following lemma which states that if a graph is β-narrow, then it has large clique or stable set (we will prove the lemma in 3.1):

(3.0.11) *Let G be a β-narrow graph. Then G has a clique or stable set of size at least $|V(G)|^{1/2\beta}$.*

Fox **[29]** proved that the 'converse' of (3.0.11) is also true: (For completeness, we will give his proof of (3.0.12) in Section 3.1)

(3.0.12) [29] *Let H be a graph that has the Erdős-Hajnal property. Then, there exists $\beta \geq 1$ such that every graph in* $\mathrm{Forb}(H)$ *is β-narrow.*

Lemmas (3.0.11) and (3.0.12) clearly imply that Conjecture 3.0.9 is equivalent to the following conjecture:

Conjecture 3.0.13. *For every graph H, there exists $\beta(H) \geq 1$ such that every $G \in \mathrm{Forb}(H)$ is $\beta(H)$-narrow.*

Main results

We are interested in establishing that the four-edge path P_4 has the Erdős-Hajnal property. In view of (3.0.11), establishing this property is equivalent to proving that every graph in $\mathrm{Forb}(P_4)$ is β-narrow for some value $\beta \geq 1$. However, dealing with graphs in $\mathrm{Forb}(P_4)$ seems quite hard. A nice property of the bull that was considered in **[12]** is the fact that the bull is self-complementary. Therefore, it is natural to consider the graphs in $\mathrm{Forb}(P_4, P_4^c)$. Using a theorem of Fouquet *et al.* **[28]**, it turns out that dealing with graphs in Forb(P_4, P_4^c) is not hard. In Section 3.3, we will prove the following:

(3.0.14) $\nu(G) \leq \log_4 5$ *for all* $G \in$ Forb(P_4^c, P_4).

The main result of this chapter deals with a larger and seemingly more complicated class of graphs:

(3.0.15) $\nu(G) \leq 3$ *for all* $G \in$ Forb(P_4^c, P_5).

Clearly, (3.0.15) together with (3.0.11) implies Theorem 3.0.10.

Organization of this chapter

This chapter is organized as follows. Section 3.1 deals with proving that the narrowness of every graph is finite, and proving the equivalence of Conjecture 3.0.9 and Conjecture 3.0.13. In Section 3.2, we describe graph decompositions that preserve narrowness. In Section 3.3, we will use these decompositions to give a simple proof that graphs in Forb(P_4^c, P_4) are ($\log_4 5$)-narrow and show that this is best possible. In Section 3.4, we start with the proof of the main result, (3.0.15), of this chapter by dealing with graphs in Forb(P_4^c, P_5) for which we additionally require that they have no induced copy of C_6, the cycle of length six. Finally, in Section 3.5 we abandon this additional requirement and finish the proof of (3.0.15).

3.1 Narrowness

We start by proving (3.0.11):

(3.0.11). *Let G be a β-narrow graph. Then G has a clique or stable set of size at least $|V(G)|^{1/2\beta}$.*

Proof. Let $\mathcal{P}$ be the set of perfect induced subgraphs of G. Let $K = \max_{P \in \mathcal{P}} |V(P)|$. Consider the function $g : V(G) \to \mathbb{R}^+$ with $g(v) = 1/K$ for all $v \in V(G)$. Clearly, $\sum_{v \in V(P)} g(v) \leq 1$ for all $P \in \mathcal{P}$. Therefore, since G is β-narrow, it follows that g satisfies

$$1 \geq \sum_{v \in V(G)} [g(v)]^\beta = \frac{|V(G)|}{K^\beta}.$$

Equivalently, we have $K \geq |V(G)|^{\frac{1}{\beta}}$. Thus, G has a perfect induced subgraph H with $|V(H)| \geq |V(G)|^{\frac{1}{\beta}}$. Since H is a perfect graph, H satisfies $|V(H)| \leq \chi(H)\alpha(H) = \omega(H)\alpha(H)$ and hence $\max(\omega(H), \alpha(H)) \geq \sqrt{|V(H)|} \geq |V(G)|^{1/2\beta}$. Therefore, H has a clique or stable set of size at least $|V(G)|^{1/2\beta}$. Since H is an induced subgraph of G, G has a clique or stable set of size at least $|V(G)|^{1/2\beta}$. This proves (3.0.11). ■

(Notice that the proof of (3.0.11) also shows that a graph G is 1-narrow if and only if G is perfect.) We need the following easy lemma.

(3.1.1) *Let $a, b \in \mathbb{R}$ and let $h : [a, b] \to \mathbb{R}$ be a convex function. Then,* $\max_{a\leq x\leq b} h(x) = \max(h(a), h(b))$.

Proof. It is trivially true that $\max_{a\leq x\leq b} h(x) \geq \max\{h(a), h(b)\}$. Since h is convex, it follows that, for all $x \in [a, b]$,

$$h(x) \leq \left(\frac{b-x}{b-a}\right) h(a) + \left(\frac{x-a}{b-a}\right) h(b) \leq \max(h(a), h(b)).$$

This proves (3.1.1). ■

Next, we give the proof of (3.0.12), the statement of which we repeat here for clarity. The argument in this proof is due to Fox **[29]**.

(3.0.12). [29] *Let H be a graph that has the Erdős-Hajnal property. Then, there exists $\beta \geq 1$ such that every graph in* Forb(H) *is β-narrow.*

Proof. Because H has the Erdős-Hajnal property, there exists $\gamma \geq 1$ such that every $G \in \text{Forb}(H)$ has a clique or stable set of size at least $|V(G)|^{1/\gamma}$. We will prove by induction that every graph in Forb(H) is β-narrow, with $\beta = \gamma + \log_2(1 + 2^\gamma)$. Let $G \in \text{Forb}(H)$ and assume inductively that every induced subgraph of G is β-narrow. Let $g : V(G) \to [0, 1]$ be a covering function for G. It suffices to show that $\sum_{v\in V(G)} [g(v)]^\beta \leq 1$.

First suppose that there exist $z \in V(G)$ such that $g(z) \geq \frac{1}{2}$. Let $\varepsilon \in [0, \frac{1}{2}]$ be such that $g(z) = 1-\varepsilon$. Let P be a perfect induced subgraph of $G|N(z)$. (3.2.2) implies that $G|(V(P) \cup \{z\})$ is a perfect induced subgraph of G. Because g is a covering function for G, it follows that $\sum_{v\in V(P)\cup\{z\}} g(v) \leq 1$ and, therefore, $\sum_{v\in V(P)} g(v) \leq \varepsilon$. Define $g' : N(z) \to [0, 1]$ by $g'(v) = g(v)/\varepsilon$. It follows from the previous that g' is a covering function for $G|N(z)$. Thus, by the inductive hypothesis,

$$\sum_{v\in N(z)} [g(v)]^\beta = \varepsilon^\beta \sum_{v\in N(z)} [g'(v)]^\beta \leq \varepsilon^\beta.$$

Similarly, $\sum_{v\in M(z)} [g(v)]^\beta \leq \varepsilon^\beta$. It follows, using (3.1.1), that

$$\begin{aligned}\sum_{v\in V(G)} [g(v)]^\beta &= [g(z)]^\beta + \sum_{v\in N(z)} [g(v)]^\beta + \sum_{v\in M(z)} [g(v)]^\beta \\ &\leq (1-\varepsilon)^\beta + 2\varepsilon^\beta \leq \max_{0\leq x\leq \frac{1}{2}} (1-x)^\beta + 2x^\beta \\ &= \max_{x\in\{0,\frac{1}{2}\}} (1-x)^\beta + 2x^\beta = \max\{1, 3\cdot 2^{-\beta}\} \leq 1.\end{aligned}$$

So we may assume that $g(v) < \frac{1}{2}$ for all $v \in V(G)$. Now, for $i = 1, 2, 3, \ldots$, let

$$A_i = \{v \in V(G) : 2^{-i-1} \leq g(v) < 2^{-i}\}.$$

Clearly, the sets A_i are disjoint and $V(G) = \bigcup_{i=1}^{\infty} A_i$. Let $i \in \mathbb{Z}_+$. Since $G|A_i \in \mathrm{Forb}(H)$, it follows that $G|A_i$ has a clique or stable set S of size at least $|A_i|^{1/\gamma}$. Because $G|S$ is perfect and g is a covering function for G, it follows, from from the fact that $g(v) \geq 2^{-i-1}$ for all $v \in S$, that

$$2^{-i-1}|A_i|^{1/\gamma} \leq 2^{-i-1}|S| \leq \sum_{v \in S} g(v) \leq 1, \text{which implies that } |A_i| \leq 2^{(i+1)\gamma}.$$

Therefore, because $g(v) < 2^{-i}$ for all $v \in A_i$, it follows that

$$\begin{aligned}\sum_{v \in V(G)} \left[g(v)\right]^{\beta} &= \sum_{i=1}^{\infty} \sum_{v \in A_i} \left[g(v)\right]^{\beta} < \sum_{i=1}^{\infty} 2^{(i+1)\gamma} 2^{-\beta i} \\ &= 2^{\gamma} \left(\frac{2^{\gamma-\beta}}{1 - 2^{\gamma-\beta}} \right) = 2^{\gamma} \left(\frac{1 + 2^{-\gamma}}{1 + 2^{\gamma}} \right) = 1,\end{aligned}$$

where we used the definition of β and the fact that $\beta > \gamma$. This proves (3.0.12). ■

Finally, we prove that every graph has finite narrowness:

(3.1.2) *Every graph on $n \geq 2$ vertices is $(\log_2 n)$-narrow.*

Proof. Let G be a graph, let $n = |V(G)|$ and let $\beta = \log_2 n$. Let g be a covering function for G. We claim that $\sum_{v \in V(G)} \left[g(v)\right]^{\beta} \leq 1$. To see this, let $z \in V(G)$ be such that $g(z)$ is maximum. If $g(z) \leq \frac{1}{2}$, then $\sum_{v \in V(G)} \left[g(v)\right]^{\beta} \leq n2^{-\beta} \leq 1$ and the claim holds. So we may assume that $g(z) > \frac{1}{2}$. Since every 2-vertex induced subgraph of G is perfect, it follows that $g(v) \leq 1 - g(z)$ for all $v \in V(G) \setminus \{z\}$. Therefore, using (3.1.1),

$$\begin{aligned}\sum_{v \in V(G)} \left[g(v)\right]^{\beta} &\leq \left[g(z)\right]^{\beta} + (n-1)\left[1 - g(z)\right]^{\beta} \leq \max_{\frac{1}{2} \leq x \leq 1} x^{\beta} + (n-1)(1-x)^{\beta} \\ &= \max((n-1)2^{-\beta}, 1) = 1.\end{aligned}$$

This proves (3.1.2). ■

3.2 Decompositions that preserve narrowness

Next, we deal with a number of graph decompositions and their relationship to the narrowness of graphs.

(3.2.1) *Let G be a graph and let $\beta \geq 1$. Suppose that for every $v \in V(G)$, either*

(i) *$G|N(v)$ is β-narrow and $G|M(v)$ is $(\beta+1)$-narrow, or*

(ii) *$G|M(v)$ is β-narrow and $G|N(v)$ is $(\beta+1)$-narrow.*

Then G is $(\beta+1)$-narrow.

Proof. Let g be a covering function for G. Choose $u \in V(G)$ with $g(u)$ maximal. We may assume that $g(u) < 1$, because every 2-vertex induced subgraph of G is perfect. Let $G_M = G|M(u)$ and $G_N = G|N(u)$. Since β-narrowness is invariant under taking complements, we may, possibly by passing to the complement, assume that G_M is $(\beta+1)$-narrow and G_N is β-narrow.

Define $f_M : V(G_M) \to \mathbb{R}^+$ by $f_M(v) = g(v)/\big[1-g(u)\big]$. Let P be a perfect induced subgraph of G_M. Since $G|(V(P) \cup \{u\})$ is perfect, it follows that $\sum_{v\in V(P)} f_M(v) \leq 1$. Since G_M is $(\beta+1)$-narrow, f_M satisfies $\sum_{v\in M}\big[f_M(v)\big]^{\beta+1} \leq 1$ and therefore

$$\sum_{v\in M}\big[g(v)\big]^{\beta+1} \leq \big[1-g(u)\big]^{\beta+1}.$$

By repeating the same argument for G_N, since G_N is β-narrow, it follows that

$$\sum_{v\in N}\big[g(v)\big]^{\beta} \leq \big[1-g(u)\big]^{\beta}.$$

Moreover, we have, by the choice of u,

$$\sum_{v\in N}\big[g(v)\big]^{\beta+1} \leq g(u)\sum_{v\in N}\big[g(v)]\big]^{\beta} \leq g(u)\big[1-g(u)\big]^{\beta}.$$

Hence, using (3.1.1),

$$\begin{aligned}\sum_{v\in V(G)}\big[g(v)\big]^{\beta+1} &= \big[g(u)\big]^{\beta+1} + \sum_{v\in M}\big[g(v)\big]^{\beta+1} + \sum_{v\in N}\big[g(v)\big]^{\beta+1}\\ &\leq \big[g(u)\big]^{\beta+1} + \big[1-g(u)\big]^{\beta+1} + g(u)\big[1-g(u)\big]^{\beta}\\ &= \big[g(u)\big]^{\beta+1} + \big[1-g(u)\big]^{\beta} \leq \max_{0\leq x\leq 1} x^{\beta+1} + (1-x)^{\beta}\\ &= \max(1,1) = 1.\end{aligned}$$

This proves (3.2.1). ■

Let G be a graph. We say that a set $Z \subseteq V(G)$ is a *homogeneous set* in G if $1 < |Z| < |V(G)|$ and $V(G) \setminus Z = A \cup C$ where A is anticomplete to Z and C is complete to Z. In this case, we say that (Z, A, C) is a *homogeneous set decomposition* of G. The following is a theorem from **[43]**.

(3.2.2) *Let G be a graph and let (Z, A, C) be a homogeneous set decomposition of G. Construct G' from $G|(A \cup C)$ by adding a vertex z that is complete to C and anticomplete to A. Let P_1 be a perfect induced subgraph of G' with $z \in V(P_1)$ and let P_2 be a perfect induced subgraph of $G|Z$. Then $G|(V(P_1) \cup V(P_2) \setminus \{z\})$ is perfect.*

It was shown in **[12]** that homogeneous set decompositions preserve β-narrowness. For our purposes, we will need a more general decomposition. We say that a set $Z \subseteq V(G)$ is a *quasi-homogeneous set* in G if there exists a partition (A, C) of $V(G) \setminus Z$ such that the following properties hold:

- $1 < |Z| < |V(G)|$.
- Z is complete to C.
- Let G' be obtained from $G|(A \cup C)$ by adding a vertex z that is anticomplete to A and complete to C. Suppose that P_1 is a perfect induced subgraph of G' with $z \in V(P_1)$ and suppose P_2 is a perfect induced subgraph of $G|X$. Then the graph $P = G|(V(P_1) \cup V(P_2) \setminus \{z\})$ is perfect.
- G contains G' as an induced subgraph.

We say that the triple (Z, A, C) is a *quasi-homogeneous set decomposition*. In the light of (3.2.2), it is easy to see that a homogeneous set decomposition is a special case of a quasi-homogeneous set decomposition. Just like homogeneous set decompositions, quasi-homogeneous sets decompositions preserve β-narrowness:

(3.2.3) *Let G be a graph and let (Z, A, C) be a quasi-homogeneous set decomposition of G. Let H_1 be the graph obtained from $G|(A \cup C)$ by adding a vertex z that is anticomplete to A and complete to C and let $H_2 = G|Z$. If H_1 and H_2 are β-narrow, then G is β-narrow.*

Proof. The proof is essentially the same as the proof of **1.3** in **[12]**, but we include it here for completeness. Let g be a covering function for G. For $i = 1, 2$, let $\mathcal{P}_i$ be the set of perfect induced subgraphs of H_i. Let $K = \max_{P \in \mathcal{P}_2} \sum_{v \in V(P)} g(v)$. Define $g_1 : V(H_1) \to \mathbb{R}^+$ as follows. For $v \in A \cup C$, let $g_1(v) = g(v)$ and let $g_1(z) = K$. Define $g_2 : V(H_2) \to \mathbb{R}^+$ by $g_2(v) = g(v)/K$ for $v \in V(H_2)$. From the definition of a quasi-homogeneous set decomposition, it follows that for every $P_1 \in \mathcal{P}_1$ with $z \in V(P_1)$ and every $P_2 \in \mathcal{P}_2$, $G|(V(P_1) \cup V(P_2) \setminus \{z\})$ is perfect. It follows that g_1 is a covering function for H_1. Since H_1 is β-narrow, it follows that

$$1 \geq \sum_{v \in V(H_1)} [g_1(v)]^\beta = \sum_{v \in A \cup C} [g(v)]^\beta + K^\beta.$$

Clearly, g_2 is a covering function for H_2. Thus, since H_2 is β-narrow, it follows that

$$1 \geq \sum_{v \in V(H_2)} [g_2(v)]^\beta = \sum_{v \in Z} \frac{[g(v)]^\beta}{K^\beta}.$$

Therefore,

$$\sum_{v \in Z} [g(v)]^\beta \leq K^\beta.$$

Finally, it follows that

$$\sum_{v \in V(G)} [g(v)]^{\beta} \leq \sum_{v \in A \cup C} [g(v)]^{\beta} + \sum_{v \in Z} [g(v)]^{\beta} \leq (1 - K^{\beta}) + K^{\beta} = 1.$$

This proves (3.2.3). ■

Let G be a graph. We say that G *admits a Σ-join* if there exist disjoint sets X_1, X_2, N_1, N_2, C, A with union $V(G)$ such that

- for $i = 1, 2$, $|X_i| \geq 2$ and X_i is a stable set, and
- for $\{i, j\} = \{1, 2\}$, X_i is complete to $C \cup N_i$ and anticomplete to $A \cup N_j$, and
- X_1 is not anticomplete to X_2.

We call $(X_1, X_2, N_1, N_2, C, A)$ a *Σ-join*. The following lemma states that Σ-joins preserve narrowness:

(3.2.4) *Let $G \in \text{Forb}(P_4^c, P_5)$ and suppose that G admits a Σ-join $(X_1, X_2, N_1, N_2, C, A)$. Let G' be obtained from $G \setminus (X_1 \cup X_2)$ by adding two adjacent vertices x_1 and x_2 such that, for $\{i, j\} = \{1, 2\}$, x_i is complete to $C \cup N_i$ and anticomplete to $A \cup N_j$. Then, $G' \in \text{Forb}(P_4^c, P_5)$ and if, for some $\beta \geq 1$, G' is β-narrow, then G is β-narrow.*

Proof. Notice first that since X_1 is not anticomplete to X_2, G contains G' as an induced subgraph and therefore $G' \in \text{Forb}(P_4^c, P_5)$. Now suppose that G' is β-narrow for some $\beta \geq 1$. For an induced subgraph P of G', let $P(X_1, X_2)$ be the graph obtained from P by substituting $G|X_i$ for x_i if $x_i \in V(P)$ (for $i = 1, 2$). We first claim the following:

($*$) *If P is a perfect induced subgraph of G', then $P(X_1, X_2)$ is perfect induced subgraph of G.*

Write $P' = P(X_1, X_2)$. Clearly, P' is an induced subgraph of G, and thus it suffices to show that P' is perfect. For suppose not. Then, P' contains either a cycle of odd length $k \geq 5$ or the complement of a cycle of length $k \geq 5$ as an induced subgraph. Since P' is an induced subgraph of G, it follows that $P' \in \text{Forb}(P_4^c, P_5)$ and, thus, P' contains no cycle of odd length at least seven and no complement of a cycle of odd length at least seven as an induced subgraph. Thus, P' has an induced cycle of length five, say $F = f_1\text{-}f_2\text{-}\cdots\text{-}f_5\text{-}f_1$. If $V(F) \cap X_1 = \emptyset$, then P' is an induced subgraph of $G \setminus X_1$, X_2 is a homogeneous set in G and hence P' is perfect by (3.2.2). Thus, we may assume that $V(F) \cap X_i \neq \emptyset$ for $i = 1, 2$. We may assume that $f_1 \in X_1$, and either $f_2 \in X_2$ or $f_3 \in X_2$. First suppose that $f_3 \in X_2$. Because f_2 is complete to $\{f_1, f_3\}$, it follows from the definition of the Σ-join that $f_2 \in C$. Because f_4 is anticomplete to $\{f_1, f_2\}$ and adjacent to f_3, it follows that $f_4 \in N_2$ and, symmetrically, $f_5 \in N_1$. But now, $x_1\text{-}f_4\text{-}f_2\text{-}f_5\text{-}x_2$ is an induced four-edge antipath in G', a contradiction. This proves that $f_3 \notin X_2$ and hence $f_2 \in X_2$. We may also assume that no two nonadjacent $f, f' \in V(F)$ satisfy $f \in X_1$ and $f' \in X_2$. Therefore, since f_4 is anticomplete to $\{f_1, f_2\}$, it follows that $f_4 \in A$. This implies that $f_3 \in N_2$ and $f_5 \in N_1$. But now, $x_1\text{-}x_2\text{-}f_3\text{-}f_4\text{-}f_5\text{-}x_1$ is an induced cycle of length five in P, contrary to the fact that P is perfect. This proves ($*$). □

To prove that G is β-narrow, let $g : V(G) \to \mathbb{Z}_+$ be a covering function for G. Define $g' : V(G') \to \mathbb{Z}_+$ as follows: for $i = 1, 2$, $g'(x_i) = \sum_{v \in X_i} g(v)$, and $g'(v) = g(v)$ for all $v \in V(G') \setminus \{x_1, x_2\}$. We claim that g' is a covering function for G'. For let P be a perfect induced subgraph of G'. Since $P(X_1, X_2)$ is a perfect induced subgraph of G by **(∗)**, it follows that

$$\begin{aligned}\sum_{v\in V(P)} g'(v) &= \sum_{\substack{i\in\{1,2\}:\\ x_i\in V(P)}} g'(x_i) + \sum_{\substack{v\in V(P)\\ v\neq x_1,x_2}} g'(v)\\ &= \sum_{\substack{i\in\{1,2\}:\\ x_i\in V(P)}} \sum_{v\in X_i} g(v) + \sum_{\substack{v\in V(P)\\ v\neq x_1,x_2}} g(v) = \sum_{v\in V(P(X_1,X_2))} g(v) \leq 1.\end{aligned}$$

This proves that g' is a covering function for G'. Since G' is β-narrow, it follows that

$$\begin{aligned}\sum_{v\in V(G)} \left[g(v)\right]^{\beta} &\leq \left[\sum_{v\in X_1} g(v)\right]^{\beta} + \left[\sum_{v\in X_2} g(v)\right]^{\beta} + \sum_{v\in V(G)\setminus(X_1\cup X_2)} \left[g(v)\right]^{\beta}\\ &= \sum_{v\in V(G')} \left[g'(v)\right]^{\beta} \leq 1,\end{aligned}$$

where we have use the fact that for $x, y \geq 0$ and $\beta \geq 1$, $x^\beta + y^\beta \leq (x+y)^\beta$. This proves that G is β-narrow, thereby proving (3.2.4). ■

3.3 Graphs in Forb($\mathbf{P_4}, \mathbf{P_4^c}$)

The following theorem is a corollary of Theorem 3.1 in **[28]**:

Theorem 3.3.1. [28] *Let $G \in$ Forb(P_4, P_4^c). If G contains an induced cycle of length five, then either G is a cycle of length five, or G has a homogeneous set.*

Clearly, by the Strong Perfect Graph Theorem **[17]**, every graph in Forb(P_4, P_4^c) that has no induced cycle of length five is perfect and thus has narrowness equal to one. It therefore suffices to consider graphs that do contain a hole of length five. Since homogeneous set decompositions preserve the narrowness of a graph, the narrowness actually only depends on the narrowness of the cycle of length five. It turns out that we can compute the narrowness of odd cycles precisely:

(3.3.2) *For odd $k \geq 5$, $\nu(C_k) = \log_{k-1} k$.*

Proof. Let $\mathcal{P}$ be the set of maximal perfect induced subgraphs of G. Observe that $\nu(G)$ is the smallest value $\beta \geq 1$ such that the following optimization problem has optimal value at most one:

$$\begin{aligned}
&\max_{\{g(v)\}_{v\in V(G)}} \sum_{v\in V(G)} \left[g(v)\right]^{\beta} && \\
&\text{subject to} \sum_{v\in V(P)} g(v) \leq 1, && \text{for all } P \in \mathcal{P} \\
&\qquad g(v) \geq 0, && \text{for all } v \in V(G).
\end{aligned}$$

This problem involves maximizing a convex function over bounded polyhedron and, thus, there exists an optimal solution that is an extreme point of the polyhedron. Therefore, the optimal value of this optimization problem is given by the maximum objective value over the extreme points of the corresponding polyhedron.

Consider the optimization problem when $G = C_k$, with $k \geq 5$ odd. Let $\{v_1, v_2, \dots, v_k\}$ denote the vertex set of C_k. Since every induced subgraph of C_k that has strictly less than k vertices is perfect, the optimization problem reads as follows:

$$\begin{aligned}
z^* = &\max_{\{g(v_i)\}_{i=1}^{k}} \sum_{i=1}^{k} \left[g(v_i)\right]^{\beta} && && \\
&\text{subject to} \sum_{i=1}^{k} g(v_i) - g(v_j) \leq 1, && \text{for all } j \in [k] && (3.1) \\
&\qquad g(v_i) \geq 0, && \text{for all } i \in [k]. && (3.2)
\end{aligned}$$

We claim that $z^* = k\left[\frac{1}{k-1}\right]^{\beta}$. Let $P \subseteq \mathbb{R}^k$ be the polyhedron defined by the inequalities in (3.1) and (3.2). As remarked previously, the optimization problem has an optimal solution that is an extreme point of P. Hence, let g be an extreme point of P. Since g is an extreme point of a polyhedron that is a subset of $\mathbb{R}^k$, there are exactly k linearly independent active constraints at g. First assume that one of the nonnegativity constraints (3.2) is active, say $g(v_i) = 0$. Then the constraint (3.1) with $j = i$ implies that $\sum_{v\in V(G)}\left[g(v)\right]^{\beta} \leq \sum_{v\in V(G)} g(v) \leq 1$ for all $\beta \geq 1$. Next, assume that none of the nonnegativity constraints is active at g. Since there are k active constraints at g, this implies that all of the constraints (3.1) are active. This implies that $\sum_{i=1}^{k} g(v_i) = 1 - g(v_j)$ for all $j \in [k]$ and, thus, $g(v_j) = \frac{1}{k-1}$ for all $j \in [k]$. Therefore, the objective value corresponding to g is $k\left[\frac{1}{k-1}\right]^{\beta} \geq 1$. Since this extreme point g clearly attains the maximum objective value over all extreme points of P, this proves that $z^* = k\left[\frac{1}{k-1}\right]^{\beta}$.

Now observe that

$$z^* = k\left[\frac{1}{k-1}\right]^{\beta} \leq 1 \iff \beta \geq \log_{k-1} k.$$

Therefore, C_k is β-narrow if and only if $\beta \geq \log_{k-1} k$. This proves (3.3.2). ∎

This puts us in a position to prove (3.0.14):

(3.0.14). $\nu(G) \leq \log_4 5$ *for every* $G \in \text{Forb}(P_4, P_4^c)$.

Proof. We prove this by induction on $|V(G)|$. Let $G \in \text{Forb}(P_4, P_4^c)$. If G is perfect, then G satisfies $\nu(G) = 1$ and we are done. So we may assume that G is not perfect. By the Strong Perfect Graph Theorem **[17]** and the fact that $G \in \text{Forb}(P_4, P_4^c)$, this implies that G has an induced cycle of length five. Thus, it follows from Theorem 3.3.1 that either G is a cycle of length five, or G admits a homogeneous set decomposition. If G is a cycle of length five, then it follows from (3.3.2) that $\nu(G) = \log_4 5$ and we are done. So we may assume that G admits a homogeneous set decomposition. Now it follows from the inductive hypothesis and (3.2.3) that $\nu(G) \leq \log_4 5$. This proves (3.0.14). ∎

Together with the fact that $C_5 \in \text{Forb}(P_4, P_4^c)$ and $\nu(C_5) = \log_4 5$, this shows that the bound found in (3.0.14) is tight. Also notice that (3.0.11) implies that every graph in $\text{Forb}(P_4, P_4^c)$ has a stable set or a clique of size at least $n^{1/2\log_4 5} = n^{\frac{1}{2}\log_5 4}$. In particular, consider C_5. The given bound states that C_5 has a stable set or a clique of size at least $5^{\frac{1}{2}\log_5 4} = 2$, which is clearly best possible.

3.4 Graphs in $\mathbf{Forb(P_4^c, P_5, C_6)}$

We start by additionally excluding the cycle of length six, C_6. Throughout the chapter, we will call an induced subgraph of a graph G that is a cycle of length k a *k-gon* in G. We will often denote the vertices of a k-gon H by, for example, $h_1, h_2, \dots, h_k$ in order. Any arithmetic involving the subscripts of these vertices is modulo k. For a k-gon H, we say that $v \in V(G) \setminus V(H)$ is a *center* for H, if v is complete to $V(H)$. Analogously, v is an *anticenter* for H if v is anticomplete to $V(H)$.

We say that a graph $G \in \text{Forb}(P_4^c, P_5, C_6)$ is a *composite graph* if there exist a 5-gon B in G and $a, c \in V(G) \setminus V(B)$ such that a is an anticenter for B and c is a center for B. We say that any graph in $G \in \text{Forb}(P_4^c, P_5, C_6)$ is *basic* if it is not composite.

This section is organized as follows. We will first prove some basic properties of graphs in $\text{Forb}(P_4^c, P_5, C_6)$. Next, we will show that composite graphs admit a quasi-homogeneous set decomposition. Finally, we will show that basic graphs satisfy the assumptions of (3.2.1) with $\beta = 1$. This will imply that all graphs in $\text{Forb}(P_4^c, P_5, C_6)$ are 2-narrow.

3.4.1 Elementary properties

We will repeatedly use the following lemmas:

(3.4.1) *Let* $G \in \text{Forb}(P_4^c)$ *and let* f_1-f_2-f_3-f_4 *be an induced path. Then no vertex is complete to* $\{f_1, f_2, f_4\}$ *and nonadjacent to* f_3.

Proof. Suppose for a contradiction that x is adjacent to f_1, f_2, and f_4 and not to f_3. Then x-f_3-f_1-f_4-f_2 is a four-edge antipath, a contradiction. This proves (3.4.1). ∎

For a 5-gon H in a graph G, we call a vertex $x \in V(G) \setminus V(H)$ that has a neighbor in $V(H)$ an *attachment of* H. The following lemma deals with attachments of 5-gons.

(3.4.2) *Let $G \in \text{Forb}(P_4^c, P_5)$ and let H be a 5-gon with vertices $h_1, h_2, \dots, h_5$ in order. Let $x \in V(G) \setminus V(H)$ with at least one neighbor in $V(H)$. Then, for some for some $i \in [5]$, one of the following holds:*

(1) *x is complete to $V(H)$ ("center"), or*

(2) *x is adjacent to h_i and x has no other neighbor in $V(H)$ ("leaf of type i"), or*

(3) *x is adjacent to h_{i+2}, h_{i+3} and x has no other neighbor in $V(H)$ ("hat of type i"), or*

(4) *x is a adjacent to h_{i+4}, h_{i+1}, nonadjacent to h_{i+2}, h_{i+3} and the adjacency between x and h_i is arbitrary ("clone of type i").*

Proof. If x is complete to $V(H)$, then outcome (1) holds. From this and from the symmetry, we may assume that x is adjacent to h_1 and not to h_2. First, suppose that x is adjacent to h_3. From (3.4.1) applied first to x and h_1-h_2-h_3-h_4 and then to x and h_5-h_1-h_2-h_3, it follows that x is nonadjacent to h_4 and h_5 and thus outcome (4) holds. So we may assume that x is nonadjacent to h_3. If x is adjacent to h_4, then outcome (4) holds. So we may assume that x is nonadjacent to h_4. If x is nonadjacent to h_5, then outcome (2) holds. If x is adjacent to h_5, then outcome (3) holds. This proves (3.4.2). ∎

We call an attachment x of H a *small attachment* if x is a leaf or a hat for H. Let $i \in [5]$. We call a pair of vertices (a, b) a *pyramid of type i* for H if a and b are adjacent, a is a leaf of type i, and b is a hat of type i. We say that $\{a, b\}$ is a pyramid if (a, b) or (b, a) is a pyramid. It turns out that whenever two small attachments are adjacent, they are of the same type. The following lemma deals with combinations of small attachments:

(3.4.3) *Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ and let H be a 5-gon. Suppose that u and v are small attachments of H. Then the following two statements hold:*

(a) *If u and v are adjacent, then, up to symmetry, for some $i \in [5]$, either*

  (A1) *u and v are leaves for H of type i; or*

  (A2) *u and v are hats for H of type i; or*

  (A3) *u is a leaf for H of type i, v is a hat for H of type i, and (u, v) is a pyramid of type i for H.*

(b) *If u and v are nonadjacent, then, up to symmetry, for some $i \in [5]$, either*

  (B1) *u is a leaf of type i and v is a leaf of type $j \in \{i-1, i, i+1\}$; or*

  (B2) *u is a hat of type i and v is a hat of type $j \in \{i-2, i, i+2\}$; or*

  (B3) *u is a leaf of type i and v is a hat of type $j \in \{i-2, i, i+2\}$.*

Proof. Let $h_1, h_2, \dots, h_5$ be the vertices of H in order. Since u and v are small attachments, each of u, v is either a leaf or a hat for H.

For part (a), suppose that u and v are adjacent. First assume that u is a leaf. From the symmetry, we may assume that u is a leaf of type 1 and v is either a leaf of type 1, 2 or 3, or a hat of type 1, 4 or 5. If v is a leaf of type 1, then outcome (A1) holds. If v is a hat of type 1, then outcome (A3) holds. If v is a leaf of type 2 or a hat of type 4, then u-v-h_2-h_3-h_4-h_5 is an induced five-edge path, a contradiction. If v is a leaf of type 3 or a hat of type 5, then u-v-h_3-h_4-h_5-h_1-u is an induced cycle of length six, a contradiction. This finishes the case when u is a leaf. So we may now assume that both u and v are hats. From the symmetry, we may assume that u is a hat of type 1 and v is a hat of type 1, 2 or 3. If v is a hat of type 1, then outcome (A2) holds. If v is a hat of type 2, then u-v-h_5-h_1-h_2-h_3-u is an induced cycle of length six, a contradiction. If v is a hat of type 3, then the adjacencies of v with respect to the path u-h_4-h_5-h_1 contradict (3.4.1). This proves part (a).

For part (b), suppose that u and v are nonadjacent. First assume that u is a leaf. From the symmetry, we may assume that u is of type 1 and v is either a leaf of type 1, 2 or 3, or a hat of type 1, 4, 5. If v is a leaf of type 1 or 2, then (B1) holds. If v is a leaf of type 3 or a hat of type 5, then u-h_1-h_5-h_4-h_3-v is an induced five-edge path, a contradiction. If v is a hat of type 1 or 4, then outcome (B3) holds. This finishes the case when u is a leaf. We may therefore assume that u and v are both hats for H. From the symmetry, we may assume that u is a hat of type 1 and v is a hat of type 1, 2 or 3. If v is a hat of type 1 or 3, then (B2) holds. If v is a hat of type 2, then u-h_3-h_2-h_1-h_5-v is an induced five-edge path, a contradiction. This proves part (b), thereby completing the proof of (3.4.3). ■

(3.4.4) *Let $G \in \text{Forb}(P_4^c, P_5)$. Let H be a 5-gon in G and suppose that x is a small attachment of H. Then, every neighbor $y \in V(G) \setminus V(H)$ of x is an attachment of H.*

Proof. Suppose that $y \in V(G) \setminus V(H)$ is adjacent to x but y has no neighbor in $V(H)$. Let $h_1, h_2, \dots, h_5$ be the vertices of H in order. We may assume that x is adjacent to h_1 and anticomplete to $\{h_2, h_3, h_4\}$. Now y-x-h_1-h_2-h_3-h_4 is an induced five-edge path, a contradiction. This proves (3.4.4). ■

(3.4.5) *Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ and let H be a 5-gon. Let (a, b) and (a', b') be two disjoint pyramids for H. Then (a, b) and (a', b') are pyramids of the same type.*

Proof. Let $h_1, h_2, \dots, h_5$ be the vertices of H in order. From the symmetry, we may assume that (a, b) is a pyramid of type 1 and (a', b') is a pyramid of type 1, 2 or 3. If (a', b') is of type 1, then the claim holds. If (a', b') is a pyramid of type 2, then b is a hat of type 1 for H and b' is a hat of type 2 for H, contrary to (A2) and (B2) of (3.4.3). If (a', b') is a pyramid of type 3, then a is a leaf of type 1 and a' is a leaf of type 3 for H, contrary to (A1) and (B1) of (3.4.3). This proves (3.4.5). ■

3.4.2 Composite graphs

Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ be a graph. Our goal is to produce a quasi-homogeneous set. In order to do so, we need to understand how different 5-gons interact with each other. To this end, we consider the following auxiliary graph. Let B be a 5-gon in G and let $\mathcal{W}$ be a graph with the follows properties:

(a) The vertices of $\mathcal{W}$ are 5-gons in G, and B is a vertex of $\mathcal{W}$.

(b) Two 5-gons H and H' are adjacent if and only if one of the following holds:

  (b1) $|V(H) \cap V(H')| = 4$ and $x \in V(H) \setminus V(H')$ is a clone for H'. In this case, we say that H and H' are *clone neighbors* and we call the edge HH' a *clone edge*.

  (b2) $B \in \{H, H'\}$, $|V(H) \cap V(H')| = 3$ and $\{x, y\} = V(H) \setminus V(H')$ is a pyramid for H'. In this case, we say that H and H' are *pyramid neighbors* and we call the edge HH' a *pyramid edge*.

(c) $\mathcal{W}$ is connected.

We call such a graph $\mathcal{W}$ a *C_5-structure around B in G*. Note that we do not require that *all* 5-gons in G are vertices of $\mathcal{W}$. Also note that the adjacency of two 5-gons is well-defined because property (b) is symmetric. We say that a C_5-structure $\mathcal{W}$ is *maximal* if $|V(\mathcal{W})|$ is maximal and, subject to that, $|E(\mathcal{W})|$ is maximal. Let $U(\mathcal{W}) = \bigcup_{H \in V(\mathcal{W})} V(H)$ denote the set of vertices of G that are 'covered' by $\mathcal{W}$.

Let $H \in V(\mathcal{W})$ and let $h_1, h_2, \dots, h_5$ be the vertices of H in order. Let $i \in [5]$ and let x be a clone of type i for H. We will write $H/x = G|((V(H) \setminus \{h_1\}) \cup \{x\})$ and we will say that H/x is *obtained from H by cloning h_i* and x is a clone *in the position of h_i*. For two 5-gons $F, H \in V(\mathcal{W})$, let $\operatorname{dist}(F, H)$ be the number of edges in a shortest path from F to H in $\mathcal{W}$.

Let us first prove a number of claims about C_5-structures:

(3.4.6) *Let $G \in \operatorname{Forb}(P_4^c, P_5)$ and let B be a 5-gon in G. Let $\mathcal{W}$ be a C_5-structure around B. Suppose that $H \in V(\mathcal{W})$ and $H' \in V(\mathcal{W})$ are clone neighbors. If c is a center for H, then c also a center for H'.*

Proof. Let c be a center for H. From the definition of a clone edge, it follows that $|V(H) \cap V(H')| = 4$. Since c is complete to $V(H)$, it follows that c has at least four neighbors in $V(H')$. Therefore, it follows from (3.4.2) that c is complete to $V(H')$. This proves (3.4.6). ■

(3.4.7) *Let $G \in \operatorname{Forb}(P_4^c, P_5)$ and let B be a 5-gon in G. Let $\mathcal{W}$ be a maximal C_5-structure around B. Let c be a center for some 5-gon in $V(\mathcal{W})$. Then either c is a center for every $H \in V(\mathcal{W})$ or $c \in U(\mathcal{W})$.*

Proof. If c is complete to all $H \in V(\mathcal{W})$, then the claim holds. So we may assume that c is not complete to at least one 5-gon in $V(\mathcal{W})$. Let $H_1, H_2 \in V(\mathcal{W})$ be such that c is complete to H_1 but not to H_2 and, subject to that, such that $\operatorname{dist}(H_1, H_2)$ is minimum. Clearly, since c is complete to $V(H_1)$ and not to $V(H_2)$, it follows that $H_1 \neq H_2$. Since $\operatorname{dist}(H_1, H_2)$ is minimum, it follows that H_1 and H_2 are neighbors. It follows from (3.4.6) that H_1 and H_2 are pyramid neighbors. We may write

$H_1 = h_1$-h_2-h_3-h_4-h_5-h_1 and $H_2 = h_1$-a-b-h_4-h_5-h_1. Since c is complete to $V(H_1)$, it follows that c has at least three neighbors in $V(H_2)$. Hence, since c is not complete to $V(H_2)$, it follows from (3.4.2) that c is a clone for H_2. Therefore, H_2/c is a 5-gon. From the maximality of $\mathcal{W}$, it follows that $H_2/c \in V(\mathcal{W})$ and, thus, that $c \in U(\mathcal{W})$. This proves (3.4.7). ■

(3.4.8) *Let $G \in$ Forb(P_4^c, P_5, C_6) and let B be a 5-gon in G. Let $\mathcal{W}$ be a maximal C_5-structure around B. Suppose that $H \in V(\mathcal{W})$ and $H' \in V(\mathcal{W})$ are clone neighbors and let x be such that $H' = H/x$. Let $h_1, h_2, \dots, h_5$ be the vertices of H in order. Let $i \in [5]$ and suppose that (p, q) is a pyramid of type i for H. Then either*

(1) *(p, q) is also a pyramid of type i for H', or*

(2) *x is a clone of type $j \in \{i-1, i+1\}$ for H and x is complete to $\{p, q, h_j\}$.*

Proof. Let $h_1, h_2, \dots, h_5$ be the vertices of H in order. From the symmetry, we may assume that (p, q) is a pyramid of type 1 for H and x is a clone of type 1, 2 or 3 for H. First assume that x is a clone of type 1 for H. It follows from (3.4.2) applied to q and H' that x is not adjacent to q. Therefore, q is a hat for H'. Since p is a neighbor of q, it follows from (3.4.4) that p has a neighbor in $V(H')$. It follows that p is adjacent to x. Thus, (p, q) is a pyramid for H' and outcome (1) holds. Next, assume that x is a clone of type 2 for H. Then it follows from (3.4.2) applied to x and H' that x is either complete or anticomplete to $\{p, q\}$. If x is anticomplete to $\{p, q\}$, then (p, q) is a pyramid for H' and thus outcome (1) holds. If x is complete to $\{p, q\}$, then it follows from (3.4.3) that x is adjacent to h_2. Hence, outcome (2) holds. So we may assume that x is a clone of type 3 for H. First suppose that p is adjacent to x. From (3.4.2) applied to x and the 5-gon h_1-h_2-h_3-q-p-h_1, it follows that x is anticomplete to $\{q, h_3\}$. But now the adjacencies of q with respect to h_3-h_4-x-p contradict (3.4.1). This proves that p is nonadjacent to x. But now, since p is a leaf of type 1 for H', q is a small attachment of H', and p and q are adjacent, it follows from (3.4.3) that q is a hat of type 1 for H' and (p, q) is a pyramid for H'. Hence, outcome (1) holds. This proves (3.4.8). ■

The goal in this section is to prove the following:

(3.4.9) *Let $G \in$ Forb(P_4^c, P_5, C_6) be a composite graph. Let B be a 5-gon in G and let A and C be the set of vertices that are complete and anticomplete, respectively, to $V(B)$. Let $\mathcal{W}$ be a maximal C_5-structure around B. Then $(U(\mathcal{W}), A \setminus U(\mathcal{W}), C \setminus U(\mathcal{W}))$ is a quasi-homogeneous set decomposition of G.*

As a first step in this direction, we prove the following lemma which states that $U(\mathcal{W})$ does not contain both all centers and all anticenters of B. This is useful, because in order for $U(\mathcal{W})$ to be a quasi-homogeneous set, we should have $|U(\mathcal{W})| < |V(G)|$.

(3.4.10) *Let $G \in$ Forb(P_4^c, P_5, C_6) and let B be a 5-gon with both a center and an anticenter. Let $\mathcal{W}$ be a maximal C_5-structure around B. Then $V(G) \setminus U(\mathcal{W}) \neq \emptyset$.*

Proof. We may assume that all centers and all anticenters for B are contained in $U(\mathcal{W})$.

(i) *B and every pyramid neighbor of B in $\mathcal{W}$ has a pyramid.*

We first claim that B has a pyramid. For suppose not. Let x be a center for B. Then it follows from (3.4.6) that x is a center for all $H \in V(\mathcal{W})$. In particular, for every $H \in V(\mathcal{W})$, $x \notin V(H)$. Therefore, $x \notin U(\mathcal{W})$, contrary to our assumption. Now let B' be any pyramid neighbor of B. Clearly, $\{p, q\} = V(B) \setminus V(B')$ is a pyramid for B'. This proves **(i)**. □

Now let a be an anticenter for B. We first show that:

(ii) *a is anticomplete to every pyramid (p, q) for B and a is an anticenter for every pyramid neighbor of B in $\mathcal{W}$.*

Let (p, q) be a pyramid for B. Suppose that $z \in \{p, q\}$ is adjacent to a. Since z is a small attachment of B, it follows from (3.4.4) that a has a neighbor in $V(B)$, contrary to the assumption that a is an anticenter for B. Since every pyramid neighbor H of B satisfies $V(H) \subseteq (V(B) \cup \{p', q'\})$ for some pyramid $\{p', q'\}$ for B, it follows from the previous that a is an anticenter for every pyramid neighbor of B. This proves **(ii)**. □

Since $a \in U(\mathcal{W})$ there exists a 5-gon $H^* \in V(\mathcal{W})$ such that $a \in V(H^*)$ and, subject to that, such that $\text{dist}(B, H^*)$ is minimum. Let P be a shortest path from H^* to B in $\mathcal{W}$ and write $P = H^*\text{-}H^1\text{-}H^2\text{-}\cdots\text{-}H^k$, where $H^k = B$ and $k = \text{dist}(B, H^*)$. From the definition of a C_5-structure, it follows that all edges in P are clone edges, except possibly $H^{k-1}\text{-}H^k$.

(iii) *$H^* = H^1/a$, $k \geq 2$, and H^1 is not a pyramid-neighbor of B.*

First suppose that $H^1 = B$. If H^* and B are pyramid neighbors, then it follows from **(ii)** that a is anticomplete to H^*, a contradiction. If H^* and B are clone neighbors, then, since $|V(B) \cap V(H^*)| = 4$ and a has two neighbors in $V(H^*)$, it follows that a has at least one neighbor in B, contradicting the fact that a is an anticenter for B. This proves that $H^1 \neq B$ and, thus, that $k \geq 2$. It follows from the definition of $\mathcal{W}$ that $H^*\text{-}H^1$ is a clone edge. Since $a \in V(H^*)$ and $a \notin V(H^1)$, it follows that $H^* = H^1/a$. Since a has a neighbor in $V(H^1)$, it follows from **(ii)** that H^1 is not a pyramid neighbor of B. This proves **(iii)**. □

(iv) *a is not a clone for H^i for $i \geq 2$.*

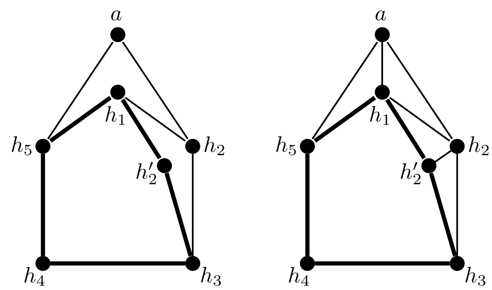
Suppose that a is a clone for H^i. Then $H^i/a\text{-}H^i\text{-}H^{i+1}\text{-}\cdots\text{-}H^k$ is a path between B and a 5-gon containing a that is shorter than P, contrary to the choice of H^*. This proves **(iv)**. □

Let $h_1, h_2, \dots, h_5$ be the vertices of H^1 in order. From the symmetry, we may assume that a is adjacent to h_2 and h_5, and possibly to h_1. Let us now consider H^2.

(v) *Up to symmetry, H^2 is obtained from H^1 by cloning h_2. Let h_2' be such that $H^2 = H^1/h_2'$. Then h_2 is nonadjacent to a, and either (see Figure 3.1)*

(1) *ah_1 and h_2h_2' are either both nonedges, or*

(2) *ah_1 and h_2h_2' are either both edges.*



(1) (2)

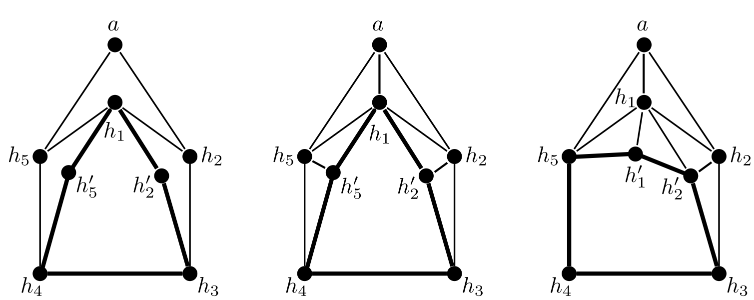
Figure 3.1: *The outcomes of* **(v)**.

Moreover, $k \geq 3$ and H^2 is not a pyramid neighbor of B.

It follows from **(iii)** that H^1 and H^2 are clone neighbors. From the symmetry, we may assume that H^2 is obtained from H^1 by cloning h_1, h_2, or h_3. It follows from **(iv)** that H^2 is not obtained from H^1 by cloning h_1. Suppose next that H^2 is obtained from H^1 by cloning h_3. Let h_3' be such that $H^2 = H^1/h_3$. It follows from (3.4.2) that a is a clone for H^2, contradicting **(iv)**. Therefore, we may assume that H^2 is obtained from H^1 by cloning h_2. Let h_2' be such that $H^2 = H^1/h_2'$. Because, from **(iv)**, a is not a clone for H^2, it follows that h_2' is nonadjacent to a. If h_2' is adjacent to h_2 and h_1 is nonadjacent to a, then h_2-h_5-h_2'-a-h_1 is an induced four-edge antipath, a contradiction. Likewise, if h_2' is nonadjacent to h_2 and h_1 is adjacent to a, then h_1-h_3-a-h_2'-h_2 is a four-edge antipath, a contradiction. This proves that ah_1 and h_2h_2' are either both edges or both nonedges.

Since a has a neighbor in H^2, it follows that $H^2 \neq B$ and hence that $k \geq 3$. Using **(ii)**, it follows that H^2 is not a pyramid neighbor of B. This proves **(v)**. □

Let H^2 and h_2' be as in **(v)**. It follows from **(v)** that we may now consider H^3, $H^3 \neq B$ and H^3 is not a pyramid neighbor of B. Therefore, H^2 and H^3 are clone neighbors.



(1) (2) (3)

Figure 3.2: *The outcomes of* **(vi)**.

(vi) *Up to symmetry, H^3 is either (see Figure 3.2)*

(1) *obtained from H^2 by cloning h_5, $h'_5 \in V(H^3) \setminus V(H^2)$ is anticomplete to $\{a, h_2, h_5\}$, and ah_1, h'_2h_2 are nonedges; or*

(2) *obtained from H^2 by cloning h_5, $h'_5 \in V(H^3) \setminus V(H^2)$ is adjacent to h_5 and anticomplete to $\{a, h_2\}$, and ah_1, h'_2h_2 are edges, or*

(3) *obtained from H^2 by cloning h_1, $h'_1 \in V(H^3) \setminus V(H^2)$ is adjacent to h_1 and anticomplete to $\{a, h_2\}$, and ah_1, h'_2h_2 are edges.*

Moreover, $k \geq 4$ and H^3 is not a pyramid neighbor of B.

Since H^2 and H^3 are clone neighbors, we may assume that H^2 is obtained from H^1 by cloning h_2. It follows from **(v)** that h'_2 is nonadjacent to a. H^3 is not obtained from H^2 by cloning h'_2, because if it is, then H^3 is adjacent to H^1, contrary to the minimality of P.

Also note that H^3 has no neighbor $H' \in V(\mathcal{W})$ such that a is a clone for H'. Because if so, then H'/a-H'-H^3-H^4-$\cdots$-H^k is a path between B and a 5-gon containing a that is shorter than P, a contradiction.

There are four cases to consider:

(a) $\underline{H^3 \text{ is obtained from } H^2 \text{ by cloning } h_1}$. (see Figure 3.3.a.) Let h'_1 be such that $H^3 = H^2/h'_1$. If h'_1 is adjacent to h_2, then H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and a is a clone for H^3/h_2, a contradiction. Therefore, h'_1 is nonadjacent to h_2. First suppose that H^2 satisfies outcome (1) of **(v)**. Since h_1-h_3-h'_1-h_2-h'_2 is not an induced four-edge antipath, it follows that h'_1 is nonadjacent to h_1. If h'_1 is nonadjacent to a, then a and h_2 are adjacent leaves of different types for H^3, contrary to (3.4.3). Therefore, h'_1 is adjacent to a. But now h'_1-h_1-a-h'_2-h_5 is an induced four-edge antipath, a contradiction. Next suppose that H^2 satisfies outcome (2) of **(v)**. From the fact that a-h'_2-h_5-h_2-h'_1 is not an induced four-edge antipath, it follows that a is nonadjacent to h'_1. It follows, from the fact that h'_2-h_5-h_2-h'_1-h_1 is not a four-edge antipath, that h'_1 is adjacent to h_1. Hence, outcome (3) holds.

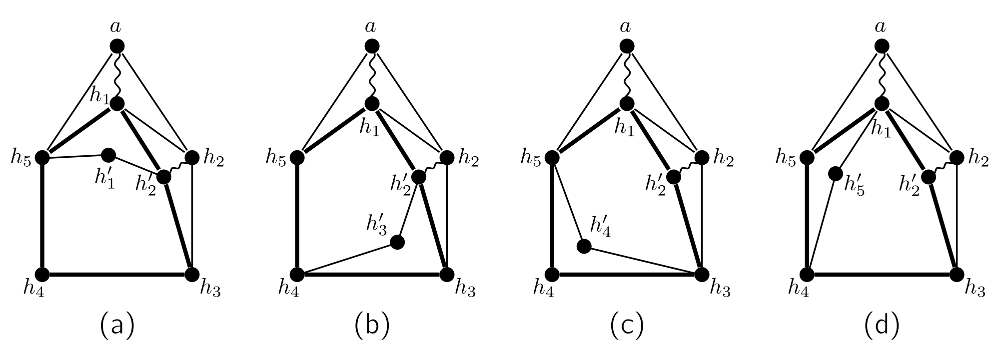


Figure 3.3: *Potential neighbors of H^2 if H^2 satisfies (1) of **(vi)**. The "wiggly" edges represent arbitrary adjacencies.*

(b) H^3 is obtained from H^2 by cloning h_3. (see Figure 3.3.b.) Let h_3' be such that $H^3 = H^2/h_3'$. Suppose that h_3' is adjacent to a. Then, it follows from (3.4.2) that a is a clone for H^3, contrary to **(iv)**. Hence, h_3' is nonadjacent to a. It follows that a is either a leaf of type 5 or a hat of type 3 for H^3. If h_3' is adjacent to h_2, then H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and a is a clone for H^3/h_3, a contradiction. Therefore, h_3' is nonadjacent to h_2 and hence h_2 is a leaf of type 1 for H^3. But now a and h_2 are small attachments of H^3 but they have different types, contrary to (3.4.3).

(c) H^3 is obtained from H^2 by cloning h_4. (see Figure 3.3.c.) Let h_4' be such that $H^3 = H^2/h_4'$. From (3.4.2) applied to h_2 and H^3, it follows that h_4' is nonadjacent to h_2 and, in particular, that h_2 is a clone for H^3. But now H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and a is a clone for H^3/h_2, a contradiction.

(d) H^3 is obtained from H^2 by cloning h_5. (see Figure 3.3.d.) Let h_5' be such that $H^3 = H^2/h_5'$. From (3.4.2) applied to h_2 and H^3, it follows that h_5' is nonadjacent to h_2 and, in particular, that h_2 is a clone for H^3. Since a is not a clone for H^3/h_2, it follows that a is nonadjacent to h_5'. If H^2 satisfies outcome (1) of **(v)**, then because h_1-a-h_5'-h_2-h_5 is not an induced four-edge antipath, it follows that h_5 is nonadjacent to h_5' and hence outcome (1) holds. If H^2 satisfies outcome (2) of **(v)**, then since h_5-h_5'-a-h_4-h_1 is not an induced four-edge antipath, it follows that h_5 is adjacent to h_5', and hence outcome (2) holds.

Now suppose that $H^3 = B$ or H^3 is a pyramid neighbor of B. Since a is an anticenter for B and for every pyramid neighbor of B, it follows that H^3 satisfies outcome (1). It follows from **(i)** and **(ii)** that H^3 has a pyramid (p, q) that is anticomplete to a. From the symmetry, we may assume that (p, q) is a pyramid of type 1, 2, or 3. First suppose that (p, q) is a pyramid of type 1 for H^3. It follows from (3.4.8) that $\{p, q\}$ is anticomplete to $\{h_2, h_5\}$. But now h_2 is a leaf for the 5-gon $F = h_1$-p-q-h_4-h_5'-h_1, a is adjacent to h_2 and a has no neighbor in F, contrary to (3.4.4). Next suppose that (p, q) is a pyramid of type 2 for H^3. Then it follows from (3.4.8) that p is nonadjacent to h_5. Hence, a is a leaf of type 5 and p is a leaf of type 2 for H^3/h_5, contrary to (3.4.3). So we may assume that (p, q) is a pyramid of type 3 for H^3. It follows from (3.4.8) that p is nonadjacent to h_5. Hence, a is a leaf of type 5 and p is a leaf of type 3 for H^3/h_5, contrary to (3.4.3). This proves that H^3 is not B or a pyramid neighbor of B and therefore that $k \geq 4$. This proves **(vi)**. □

Let H^3 be as in **(vi)**. It follows from **(vi)** that we may now consider H^4, which is a clone neighbor of H^3. Now, again, since P is a shortest path from a 5-gon that contains a to B, it follows that there is no one- or two-edge path in $\mathcal{W}$ from H^4 to a 5-gon for which a is clone.

First, suppose that H^3 satisfies outcome (1) or (2) of **(vi)**. Let h_5' be as in outcome (1) and (2) of **(vi)**. From the symmetry, we may assume that H^4 is obtained from H^3 by cloning h_1, h_2', or h_3. We need to check a number of cases:

(a) H^4 is obtained from H^3 by cloning h_1. Let h_1' be such that $H^4 = H^3/h_1'$. First suppose that

h'_1 is nonadjacent to h_2. It follows that h_2 is a leaf of type 3 or a hat of type 5 for H^4. Since a is adjacent to h_2, it follows from (3.4.4) that a is adjacent to h'_1. But now a is a leaf of type 1 for H^4 and a is adjacent to h_2, contrary to (3.4.3). Therefore, h'_1 is adjacent to h_2 and, from the symmetry, h'_1 is adjacent to h_5. But now the path H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(b) H^4 is obtained from H^3 by cloning h'_2. Now H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(c) H^4 is obtained from H^3 by cloning h_3. Let h'_3 be such that $H^4 = H^3/h'_3$. From (3.4.2) applied to h_5 and H^4, it follows that h'_3 is nonadjacent to h_5 and, in particular, that h_5 is a clone for H^4. Since a is a not a clone for H^4/h_5, it follows from (3.4.2) that a is nonadjacent to h'_3. If h'_3 is adjacent to h_2, then H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is a clone, a contradiction. Hence, h'_3 is nonadjacent to h_2 and therefore h_2 is a small attachment of H^4. Since a is adjacent to h_2, it follows from (3.4.4) that a is adjacent to h_1 and hence that outcome (2) of **(vi)** holds. But now a is a leaf of type 1 for H^4, h_2 is a hat of type 4 for H^4, and a and h_2 are adjacent, contrary to (3.4.3).

This proves that H^3 does not satisfy outcome (1) or outcome (2) of **(vi)**. So next suppose that H^3 satisfies outcome (3) of **(vi)**. We need to check a number of cases:

(a) H^4 is obtained from H^3 by cloning h'_1. H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(b) H^4 is obtained from H^3 by cloning h'_2. Let h''_2 be such that $H^4 = H^3/h''_2$. Since a is not a clone for H^4, it follows that h''_2 is nonadjacent to a. If h''_2 is adjacent to h_1, then H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction. Therefore h''_2 is nonadjacent to h_1. But now h_1 is is a hat of type 3 and a is a leaf of type 5 for H^4, and h_1 and a are adjacent, contrary to (3.4.4).

(c) H^4 is obtained from H^3 by cloning h_3. Let h'_3 be such that $H^4 = H^3/h'_3$. It follows from (3.4.2) that a is nonadjacent to h'_3. From (3.4.2) applied to h_1 and H^4, it follows that h'_3 is nonadjacent to h_1. If h'_3 is nonadjacent to h_2, then h_2 and a are leaves of type 3 and 5, respectively, and a and h_2 are adjacent, contrary to (3.4.3). Therefore, h'_3 is adjacent to h_2. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(d) H^4 is obtained from H^3 by cloning h_4. Let h'_4 be such that $H^4 = H^3/h'_4$. By (3.4.2) applied to h_1 and H^4, it follows that h'_4 is nonadjacent to h_1. By (3.4.2) applied to h_2 and H^4/h_1, it follows that h'_4 is nonadjacent to h_2. By (3.4.2) applied to a and $H^4/h_1/h_2$, it follows that h'_4 is nonadjacent to a. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(e) H^4 is obtained from H^3 by cloning h_5. Let h'_5 be such that $H^4 = H^3/h'_5$. From (3.4.2) applied to h_2 and H^4, it follows that h'_5 is nonadjacent to h_2. If h'_5 is nonadjacent to h_1, then h_1

and h_2 are hats of type 4 and 5, respectively, and h_1 and h_2 are adjacent, contrary to (3.4.3). Therefore, h_5' is adjacent to h_1. Since h_2 is a hat for H^4 and a is adjacent to h_2, it follows from (3.4.4) that a is adjacent to h_5'. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

This proves that H^3 does not satisfy any of the outcomes of **(vi)**, a contradiction. This completes the proof of (3.4.10). ■

Next, we are interested in how vertices in $V(G)\backslash U(\mathcal{W})$ can attach to $U(\mathcal{W})$ where $\mathcal{W}$ is a C_5-structure.

(3.4.11) *Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ and let B be a 5-gon. Let $\mathcal{W}$ be a maximal C_5-structure around B. Let $x \in V(G) \setminus U(\mathcal{W})$ and assume that x is not a center for $\mathcal{W}$. Let u and v be two nonadjacent neighbors of x and assume that $u \in U(\mathcal{W})$. Then, for every $H \in V(\mathcal{W})$ such that $u \in V(H)$, v is a clone for H in the same position as u and, in particular, $v \in U(\mathcal{W})$.*

Proof.

(i) *If (a, b) is a pyramid for some $H \in V(\mathcal{W})$, then $\{a, b\} \subset U(\mathcal{W})$.*

Let H^* be a 5-gon for which (a, b) is a pyramid and, subject to that, such that dist(H^*, B) is minimum. Let $h_1, h_2, \dots, h_5$ be the vertices of H^* in order. From the symmetry, we may assume that (a, b) is a pyramid of type 1 for H^*.

Let P be a shortest path from H^* to B. It follows from the definition of a maximal C_5-structure that, if $H^* = B$, then $\{a, b\} \subset U(\mathcal{W})$. So we may assume that $H^* \neq B$ and hence that $|E(P)| \geq 1$. Let H^1 be the neighbor of H^* in P. Since H^* was chosen with dist(H^*, B) minimum, it follows that $\{a, b\}$ is not a pyramid for H^1.

First suppose that H^1 is a clone neighbor of H^*. Let x be such that $H^1 = H^*/x$. From (3.4.8) and the fact that $\{a, b\}$ is not a pyramid for H^1, it follows that H^1 is obtained from H^* by cloning h_2 or h_5 and x is complete to $\{a, b\}$. But now, from the maximality of $\mathcal{W}$, H^1-H^1/b-$H^1/b/a$ is a path in $\mathcal{W}$ and hence $\{a, b\} \subset U(\mathcal{W})$.

Therefore, we may assume that H^1 is a pyramid neighbor of H^*. From the definition of a maximal C_5-structure and the fact that $H^* \neq B$, it follows that $H^1 = B$. Let $\{p, q\} = V(B)\backslash V(H^*)$. We claim that either (p, q) or (q, p) is a pyramid of type 1. If $\{p, q\} \cap \{a, b\} = \emptyset$, then, since (a, b) is a pyramid of type 1 for H^*, it follows from (3.4.5) that (p, q) or (q, p) is a pyramid of type 1 for H^*. If $\{p, q\} \cap \{a, b\} \neq \emptyset$, then it follows from the definition of a pyramid that (p, q) or (q, p) is a pyramid of type 1 for H^*. Hence, we may assume that $V(H^1) = V(B) = \{h_1, p, q, h_4, h_5\}$. This proves that (p, q) or (q, p) is a pyramid of type 1. From the symmetry, we may assume that (p, q) is a pyramid of type 1.

If $(a, b) = (p, q)$, then $\{a, b\} \subset V(B)$ and hence $\{a, b\} \subset U(\mathcal{W})$. If $a \neq p$ and $b = q$, then a is a clone for B and $b \in V(B)$ and, therefore, $\{a, b\} \subset U(\mathcal{W})$. If $a = p$ and $b \neq q$, then b is a clone for B and $a \in V(B)$ and, therefore, $\{a, b\} \subset U(\mathcal{W})$.

So we may assume that $\{a, b\} \cap \{p, q\} = \emptyset$. Now first suppose that a is adjacent to q. Then a is a clone for B and b is a clone for B/a. Hence, by maximality of $\mathcal{W}$, it follows that $B/a, B/a/b \in V(\mathcal{W})$ and, therefore, that $\{a, b\} \subset U(\mathcal{W})$. Next, suppose that b is adjacent to p. Then b is a clone for B and a is a clone for B/b. Hence, by maximality of $\mathcal{W}$, it follows that $B/b, B/b/a \in V(\mathcal{W})$ and, therefore, that $\{a, b\} \subset U(\mathcal{W})$.

It follows that we may assume that the only possible edges between $\{a, b\}$ and $\{p, q\}$ are ap and bq. It follows from (3.4.3) that exactly one of ab and pq is an edge and hence that $\{a, b\}$ is a pyramid for B. If a is adjacent to p, then (b, a) is a pyramid of type 4 for B, contrary to (3.4.5). If b is adjacent to q, then (a, b) is a pyramid of type 1 for B. By maximality of $\mathcal{W}$, it follows that $\{a, b\} \subset U(\mathcal{W})$. This proves **(i)**. □

Let $H \in V(\mathcal{W})$ such that $u \in V(H)$ and let h_1, h_2, h_3, h_4, h_5 be the vertices of H in order. From the symmetry, we may assume that $h_1 = u$. It follows from (3.4.7) and the assumption that x is not a center for $\mathcal{W}$ that x is not complete to $V(H)$. Moreover, since $\mathcal{W}$ is maximal and $x \notin U(\mathcal{W})$, it follows that x is not a clone for H. Therefore x is either a leaf or a hat for H. From the symmetry, we may assume that x is anticomplete to $\{h_2, h_3, h_4\}$, but possibly adjacent to h_5. Because x is a small attachment of H and u is adjacent to x, it follows from (3.4.4) that u has at least one neighbor in $V(H)$. Since u and v are nonadjacent, v is not complete to H. Hence, it follows from (3.4.2) that v is either a small attachment or a clone for H.

First suppose that v is a small attachment of H. Then, from (3.4.3) and the fact that u and v are nonadjacent, it follows that (x, v) is a pyramid for H. But now, by **(i)**, $\{x, v\} \subset U(\mathcal{W})$, contradicting the fact that $x \notin U(\mathcal{W})$.

So we may assume that v is a clone for H. If v is adjacent to h_2 and h_5, then the claim holds. Therefore, we may assume that v is adjacent to at most one of h_2, h_5. Since u and v are nonadjacent, it follows that v is a clone of type 3 or 4. If v is a clone of type 3, then it follows from (3.4.2) that x is a clone for H/v and hence $x \in U(\mathcal{W})$, a contradiction. If v is a clone type 4, then again x is a clone for H/v and hence $x \in U(\mathcal{W})$, a contradiction. This proves (3.4.11). ■

We are now in a position to prove (3.4.9).

Proof of (3.4.9). Let B be a 5-gon with a center and an anticenter and let $\mathcal{W}$ be a maximal C_5-structure around B. Let $Z = U(\mathcal{W})$, let C be the set of centers for $\mathcal{W}$ and let A be $V(G) \setminus (Z \cup C)$. It follows from (3.4.10) that $A \cup C \neq \emptyset$. We claim that (Z, A, C) is a quasi-homogeneous set decomposition of G. Clearly, C is complete to Z.

(i) *There exists $z \in Z$ that is anticomplete to A.*

Let $b_1, b_2, \dots, b_5$ be the vertices of B in order. Let $K_1, K_2, \dots, K_q$ be the components of $G|A$. We may assume that $V(B) \cap \bigcup_{v \in A} N(v) = V(B)$, because otherwise the claim holds. It follows from (3.4.3) and the maximality of $U(\mathcal{W})$ that, for $j \in [q]$, every two vertices $u, v \in V(K_j)$ are either leaves of the same type or hats of the same type with respect to B. In particular, for each $j \in [q]$, $V(B) \cap N(u) = V(B) \cap N(v)$ for all $u, v \in V(K_j)$. Since $V(B) \cap \bigcup_{v \in A} N(v) = V(B)$, it follows that there exist a stable set $S \subseteq A$ such that $V(B) \cap \bigcup_{v \in S} N(v) = V(B)$. First suppose that some $s_1 \in S$ is a leaf for B. From the symmetry, we may assume that $V(B) \cap N(s_1) = b_1$. For $i = 2, 5$, let $s_i \in S$ be a neighbor of b_i. It follows from (3.4.3) applied to s_1 and s_2 that s_2 is either a leaf of type 2 for B, or a hat of type 4. This implies that $V(B) \cap N(s_2) \subseteq \{b_1, b_2\}$ and, symmetrically, $V(B) \cap N(s_5) = \{b_1, b_5\}$. But now, s_2-b_2-b_3-b_4-b_5-s_5 is an induced five-edge path, a contradiction. Thus, we may assume that every vertex in S is a hat for B. Let $s_1 \in S$. From the symmetry, we may assume that s_1 is a hat of type 1 for B. For $i = 2, 5$, let $s_i \in S$ be a neighbor of b_i. It follows from (3.4.3) applied to s_1 and s_2 that s_2 is a hat of type 4 for B. Symmetrically, s_5 is a hat of type 3 for B. But now, s_2 and s_5 contradict (3.4.3). This proves **(i)**. □

Let G' be as in the definition of the quasi-homogeneous set decomposition. It follows from **(i)** that there exists $z \in Z$ that is complete to C and anticomplete to A. Therefore, G contains G' as an induced subgraph. Let P be as in the definition of a quasi-homogeneous set decomposition. Suppose that P is not perfect. Since P is an induced subgraph of G, it does not have an induced four-edge antipath or an induced five-edge path. It follows that P contains an induced cycle F of length five. Let $f_1, f_2, \dots, f_5$ be the vertices of F in order.

(ii) *No edge of F has one endpoint in Z and one endpoint in C.*

From the symmetry, we may assume that $f_1 \in Z$ and $f_2 \in C$. Since C is complete to Z, and f_4 is nonadjacent to f_1 and f_2, it follows that $f_4 \in A$. Moreover, since f_5 is nonadjacent to f_2, it follows for the same reason that $f_5 \in A \cup C$. If $f_5 \in A$, then (3.4.11) with $x = f_5$, $u = f_1$ and $v = f_4$, implies that $f_4 \in Z$, a contradiction. Therefore, we may assume that $f_5 \in C$. Because f_3 is nonadjacent to f_1 and f_5, it follows that $f_3 \notin C \cup Z$, and hence that $f_3 \in A$. But now z-f_2-f_3-f_4-f_5-z is an induced cycle of length five in P_1, contradicting the fact that P_1 is perfect. This proves **(ii)**. □

Let P^* be obtained from P by deleting all edges between $A \cap V(P)$ and $Z \cap V(P)$. It follows from (3.2.2) that P^* is perfect. Therefore, F is not an induced subgraph of P^*. It follows that there exist two vertices $a \in Z$ and $b \in A$ that are adjacent in G such that $a, b \in V(F)$, say $f_1 = a$ and $f_2 = b$.

Let $H \in V(\mathcal{W})$ be such that $f_1 \in V(H)$. Let $h_1, h_2, \dots, h_5$ be the vertices of H in order. We may assume that $f_1 = h_1$.

(iii) *No vertex $w \in A$ is a clone or a center for H.*

If w is a clone for H, then it follows from the maximality of $\mathcal{W}$ that $w \in Z$, a contradiction. If w is a center for H, then it follows from (3.4.7) that $w \in Z \cup C$, a contradiction. This proves **(iii)**. □

(iv) *f_3 is a clone of type 1 for H and $\{f_3, f_4, f_5\} \subset Z$.*

Since f_1 is nonadjacent to f_3, it follows from (3.4.11) that $f_3 \in Z$ and f_3 is a clone in the same position as f_1 for H. It follows from **(ii)** that $f_5 \in A \cup Z$. Suppose that $f_5 \in A$. Since f_4 is nonadjacent to f_1, it follows from (3.4.11) that f_4 is also a clone of type 1 for H. If f_5 is adjacent to both h_5 and h_2, then it follows from (3.4.2) that f_5 is a clone or a center for

H, contrary to **(iii)**. Therefore, from the symmetry, we may assume that f_5 is nonadjacent to h_2. But now h_2-f_5-f_3-f_1-f_4 is an induced four-edge antipath, a contradiction. This proves that $f_5 \in Z$ and, from the symmetry, that $f_4 \in Z$, and hence this proves **(iv)**. $\square$

Since f_5 is adjacent to f_1, but not to f_3, it follows that $f_5 \notin V(H)$. Since f_4 is adjacent to f_3 but not to f_1, it follows that $f_4 \notin V(H)$. It follows from **(iii)** that f_2 is not a clone or a center for H and hence that f_2 is nonadjacent to h_3 and h_4.

We claim that $\{f_4, f_5\}$ anticomplete to $\{h_2, h_5\}$. For suppose not. From the symmetry, we may assume that f_4 is adjacent to h_2. If f_4 is nonadjacent to h_5, then f_3-f_1-f_4-h_5-h_2 is an induced four-edge antipath, a contradiction. Therefore, f_4 is adjacent to h_5. If f_2 is adjacent to both h_2 and h_5, then it follows from (3.4.2) that f_2 is a clone or a center for H, contrary to **(iii)**. Hence, from the symmetry, we may assume that f_2 is nonadjacent to h_2. But now f_3-f_1-f_4-f_2-h_2 is an induced four-edge antipath, a contradiction. This proves that $\{f_4, f_5\}$ anticomplete to $\{h_2, h_5\}$.

It follows from (3.4.4) applied to h_3, h_4 and h_2-f_3-f_4-f_5-f_1-h_2 that there is at least one edge between $\{h_3, h_4\}$ and $\{f_4, f_5\}$. From the symmetry, we may assume that f_5 is adjacent to h_4. It follows from (3.4.2) applied to h_4 and h_5-f_3-f_4-f_5-f_1-h_5 that h_4 is nonadjacent to f_4. It follows from (3.4.2) applied to f_5 and H that f_5 is nonadjacent to h_3. By applying (3.4.4) to h_4, h_3 and F, h_3 has a neighbor in $V(F)$. Therefore, h_3 is adjacent to f_4. But now h_3 and h_4 are adjacent leaves for F that have different types, contradicting (3.4.3). This proves (3.4.9). ■

3.4.3 Basic graphs

In the previous section, we showed that composite graphs in $\text{Forb}(P_4^c, P_5, C_6)$, *i.e.*, graphs that have a 5-gon with both a center and an anticenter, admit a quasi-homogeneous set decomposition. In this section, we will analyze basic graphs. It turns out that if a graph does not contain a 5-gon with both a center and an anticenter, then a 'dual' statement is also true: there is a no vertex that simultaneously serves as a center for some 5-gon in G and as an anticenter for some other 5-gon in G (we will prove this shortly). In particular, this implies that for every $v \in V(G)$, either $G|N(v)$ or $G|M(v)$ is perfect (and, equivalently, 1-narrow).

(3.4.12) *Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ and suppose that no 5-gon has both a center and an anticenter. Then there do not exist v, A and B such that $v \in V(G)$, A and B are* 5*-gons in B, and v is a center for A and an anticenter for B.*

Proof. Suppose that v is a center for a 5-gon A and an anticenter for a 5-gon B. Since v is complete to $V(A)$ and anticomplete to $V(B)$, it follows that $V(A) \cap V(B) = \emptyset$. Let $a_1, a_2, \dots, a_5$ and $b_1, b_2, \dots, b_5$ be the vertices of A and B, respectively, in order.

(i) *Every $x \in V(B)$ is a small attachment of A and all $x \in V(B)$ are of the same type.*

It follows from (3.4.2) that x is either an anticenter, or a small attachment, or a clone, or a center for A. Since G is basic, A does not have an anticenter and, hence, x is not an anticenter for A. Now suppose that x is a clone for A. It follows from (3.4.2) that v is adjacent to x, contradicting the fact that v is anticomplete to $V(B)$. This proves that every vertex in $V(B)$ is either a small attachment or a center for A.

Suppose that some vertex in $V(B)$ is complete to $V(A)$. Since B has no center, not all vertices in $V(B)$ are centers for A. Therefore, there are adjacent $y, z \in V(B)$ such that y is complete to $V(A)$ and z is not. Therefore, z is a small attachment of A. Let $a \in V(A)$ be a neighbor of z and let $a' \in V(A)$ be a nonneighbor of a. Since z is a small attachment of A, it follows that a' is nonadjacent to z. But now a-a'-z-v-y is an induced four-edge path, a contradiction. This proves that every vertex in $V(B)$ is a small attachment of A. Now suppose that not all vertices of $V(B)$ are of the same type with respect to A. Then there exist adjacent $b, b' \in V(B)$ such that b and b' are small attachments for A, but of different types, contradicting (3.4.3). This proves **(i)**. □

(ii) *Let $x \in V(A)$. Then x is either a clone or an anticenter for B.*

Suppose that x is not a clone or an anticenter for B. Since G is basic, B does not have a center and, hence, x is not complete to $V(B)$. Then it follows from (3.4.2) that x is a small attachment of B. But now v is a neighbor of a small attachment of B and v has no neighbor in $V(B)$, contrary to (3.4.4). This proves **(ii)**. □

From **(i)** and the symmetry, we may assume that all $b \in V(B)$ are of type 1. That is, for every $b \in V(B)$, b is either adjacent to a_1 and anticomplete to $\{a_2, a_3, a_4, a_5\}$, or b is adjacent to a_3 and a_4 and anticomplete to $\{a_1, a_2, a_5\}$. Since B does not have a center, at least one of the vertices of B is a leaf and at least one of them is a hat. From the symmetry, we may assume that b_1 is a leaf for A that is adjacent to a_1. Since from **(ii)** every vertex of A is either a clone or an anticenter for B, it follows that we may assume that a_1 is adjacent to b_4 and a_1 is anticomplete to $\{b_2, b_3\}$. Since a_1 is anticomplete to $\{b_2, b_3\}$, it follows from **(i)** that b_2 and b_3 are complete to $\{a_3, a_4\}$. Because b_1 and b_4 are leaves, it follows that $\{b_1, b_4\}$ is anticomplete to $\{a_3, a_4\}$. Therefore, it follows from (3.4.2) applied to a_3 and B that a_3 is a hat for B, contradicting **(ii)**. This proves (3.4.12). ■

We can now prove that

Theorem 3.4.13. *Every graph $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ is 2-narrow.*

Proof. We prove this by induction on $|V(G)|$. If G is perfect, then G is 1-narrow and there is nothing to prove. So we may assume that G is not perfect. From the fact that G has no induced four-edge antipath and no induced five-edge path, it follows that G contains a 5-gon. First suppose that G contains a 5-gon with a center and an anticenter. Then, by (3.4.9), G admits a quasi-homogeneous set decomposition (Z, A, C). Let G' be the graph obtained from $G|(A \cup C)$ by adding a vertex z that is anticomplete to A and complete to C. By the induction hypothesis, G' and $G|Z$ are 2-narrow. It follows from (3.2.3) that G is 2-narrow. So we may assume that G has no 5-gon that has both a center and an anticenter. Let $v \in V(G)$. It follows from the induction hypothesis that $G|N(v)$ and $G|M(v)$ are both 2-narrow. Moreover, it follows from (3.4.12) that either $G|N(v)$ or $G|M(v)$ is perfect and hence 1-narrow. Since this is true for every $v \in V(G)$, it follows from (3.2.1) that G is 2-narrow. This proves Theorem 3.4.13. ■

3.5 Graphs in Forb(P_4^c, P_5)

In this section, we will prove that every graph in Forb(P_4^c, P_5) is 3-narrow. Let $G \in$ Forb(P_4^c, P_5) and suppose that G does not contain a 6-gon with a center. Then it follows that $G|N(v) \in$ Forb(P_4^c, P_5, C_6) for every $v \in V(G)$. In the previous section, we proved that therefore $G|N(v)$ is 2-narrow for every $v \in V(G)$. Now we may apply (3.2.1) to conclude that G is 3-narrow (for details, see the proof of (3.0.15) at the end of this section). The remaining case is when G does contain a 6-gon with a center. We deal with this case in (3.5.2). We will start with a lemma that deals with attachments of 6-gons.

(3.5.1) *Let $G \in$ Forb(P_4^c, P_5) and let H be a 6-gon in G with vertices $h_1, h_2, \dots, h_6$ in order. Let $v \in V(G) \setminus V(H)$ and suppose that v has a neighbor and a nonneighbor in $V(H)$. Then, up to symmetry, either*

- (x) *v is complete to $\{h_1, h_3, h_5\}$ and v is anticomplete to $\{h_2, h_4, h_6\}$, or*
- (y) *v is complete to $\{h_3, h_6\}$, v is anticomplete to $\{h_1, h_2\}$ and v is either complete or anticomplete to $\{h_4, h_5\}$, or*
- (z) *v is complete to $\{h_1, h_3\}$, anticomplete to $\{h_4, h_5, h_6\}$, and the adjacency between v and h_2 is arbitrary.*

Proof. We may assume that v is adjacent to h_1 and nonadjacent to h_2. Suppose that v is adjacent to h_3. Since h_1-h_2-h_3-h_4 is an induced path, and v is complete to $\{h_1, h_3\}$ and nonadjacent to h_2, it follows from (3.4.1) that v is nonadjacent to h_4. From the symmetry, it follows that v is nonadjacent to h_6. If v is adjacent to h_5, then (x) holds. If v is nonadjacent to h_5, then (z) holds. So we may assume that v is nonadjacent to h_3. If v is nonadjacent to h_4, then, since v-h_1-h_2-h_3-h_4-h_5 is not an induced five-edge path, it follows that v is adjacent to h_5 and (z) holds. So we may assume that v is adjacent to h_4. Because h_4-h_5-h_6-h_1 is an induced path and v is adjacent to h_1 and h_4, it follows from (3.4.1) that v is either complete or anticomplete to $\{h_5, h_6\}$. Therefore, (y) holds. This proves (3.5.1). ■

Let $G \in$ Forb(P_4^c, P_5) and let H be a 6-gon in G. We call a vertex $v \in V(G) \setminus V(H)$ an *(x)-vertex, (y)-vertex, or (z)-vertex for H* if v satisfies (x), (y), or (z) of (3.5.1), respectively. Let $z \in V(G) \setminus V(H)$ be a (z)-vertex for H. Then, there exists a unique vertex $h \in V(H)$ such that $H' = G|((V(H) \setminus \{h\}) \cup \{z\})$ is a 6-gon. We say that H' is *the 6-gon obtained from rerouting H through z*.

(3.5.2) *Let $G \in$ Forb(P_4^c, P_5) and suppose that G contains a 6-gon with a center. Then G admits either a quasi-homogeneous set decomposition, or a Σ-join.*

Proof. Let H be a 6-gon with a center and let $h_1, h_2, \dots, h_6$ be the vertices of H in order. Let C be the set of vertices that are complete to $V(H)$. Notice that $C \neq \emptyset$. Let X, Y, and Z be the sets of (x)-vertices, (y)-vertices, and (z)-vertices for H, respectively.

(i) *C is complete to $X \cup Y \cup Z$.*

Let $c \in C$ and $z \in Z$. Let H' be the 6-gon obtained from rerouting H through z. Then c has at least five neighbors in $V(H')$ and, hence, (3.5.1) implies that c is adjacent to z. This proves that C is complete to Z. Now let $x \in X$. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$ and anticomplete to $\{h_2, h_4, h_6\}$. Since h_6-h_1-x-h_3 is an induced path and c is complete to $\{h_1, h_3, h_6\}$, it follows from (3.4.1) that c is adjacent to x. Hence, C is complete to X. Next, let $y \in Y$. We may assume that y is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Then h_1-h_6-y-h_3 is an induced path and c is complete to $\{h_1, h_3, h_6\}$. It follows from (3.4.1) that y is adjacent to c and hence that Y is complete to C. This proves **(i)**. □

Let Y' be the set of vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z)$ with a neighbor in Y. Let X' be the set of vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y')$ with a neighbor in X. Let X'' be the set of the vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y' \cup X')$ with a neighbor in X'. Let $A = V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y' \cup X' \cup X'')$. Since $(A \cup X' \cup X'' \cup Y') \cap (X \cup Y \cup Z \cup C) = \emptyset$, (3.5.1) implies that $A \cup Y' \cup X' \cup X''$ is anticomplete to $V(H)$. It follows from the definition of Y', X', X'', and A that $X' \cup X'' \cup A$ is anticomplete to Y, X is anticomplete to $X'' \cup A$, and X' is anticomplete to A.

(ii) *Z is anticomplete to $A \cup X' \cup X'' \cup Y'$, Y' is anticomplete to $A \cup X' \cup X''$, A is anticomplete to X'', and X is anticomplete to Y.*

First, suppose that $z \in Z$ is adjacent to $a \in A \cup X' \cup X'' \cup Y'$. Let H' be obtained from rerouting H through z. Then it follows that a has exactly one neighbor in $V(H')$, contrary to (3.5.1). This proves that Z is anticomplete to $A \cup X' \cup X'' \cup Y'$.

Next, suppose that $y' \in Y'$ is adjacent to $a \in A \cup X' \cup X''$. Let $y \in Y$ be a neighbor of y'. We may assume that y is adjacent to h_3 and not to h_1 and h_2. Now h_1-h_2-h_3-y-y'-a is an induced five-edge path, a contradiction. This proves that Y' is anticomplete to $A \cup X' \cup X''$.

Next, suppose that $x'' \in X''$ is adjacent to $a \in A$. Then let $x' \in X'$ be a neighbor of x'' and let $x \in X$ be a neighbor of x'. From the symmetry, we may assume that x is adjacent to h_1 and not to h_2. Then h_2-h_1-x-x'-x''-a is an induced five-edge path, a contradiction. This proves that A is anticomplete to X''.

Finally, suppose that $x \in X$ and $y \in Y$ are adjacent. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$ and anticomplete to $\{h_2, h_4, h_6\}$, and that y is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Now, h_1-h_2-h_3-y is an induced path, x is complete to $\{h_1, h_3, y\}$ and x is nonadjacent to h_2, contrary to (3.4.1). This proves **(ii)**. □

The following two claims deal with the case when $Y \neq \emptyset$.

(iii) *Suppose that $Y \neq \emptyset$. Then there do not exist x, p, q such that $x \in X \cup Y$, $p, q \in X' \cup X'' \cup Y'$, and x-p-q is an induced path.*

Suppose that $Y \neq \emptyset$ and suppose that such x, p, q exist. First suppose that $x \in Y$. We may assume that x is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Now h_1-h_2-h_3-x-p-q is an induced five-edge path, a contradiction. We may therefore assume that $x \in X$. Let $y \in Y$. It follows from **(ii)** that y is nonadjacent to x. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$, y is complete to $\{h_3, h_6\}$ and y is anticomplete to $\{h_1, h_2\}$. Since q-p-x-h_1-h_6-y is not an induced five-edge path, it follows that y is adjacent to at least one of p and q. Because we already proved that no vertex in Y forms a two-edge induced path with p and q, it follows that y is complete to $\{p, q\}$. But now x-h_3-y-q is an induced path, p is complete to $\{x, y, q\}$, and p is nonadjacent to h_3, contrary to (3.4.1). This proves **(iii)**. □

(iv) *If $Y \neq \emptyset$, then the lemma holds.*

Suppose that $Y \neq \emptyset$. We claim that $X'' = \emptyset$. For suppose that $x'' \in X''$. Then let $x' \in X'$ be a neighbor of x'', and let $x \in X$ be a neighbor of x'. Then x-x'-x'' is an induced path with $x \in X$ and $x', x'' \in X' \cup X''$, contrary to **(iii)**. This proves that $X'' = \emptyset$.

Let A' be the union of all the components K of $G|(X' \cup Y')$ such that C is not complete to K. Let $N = A \cup A'$ and $U = (V(H) \cup X \cup Y \cup Z \cup X' \cup Y') \setminus A'$. We claim that (U, N, C) is a quasi-homogeneous set decomposition of G. Let G' be as in the definition of the quasi-homogeneous set decomposition. First notice that any vertex in $V(H)$ is complete to C and anticomplete to N, and therefore G contains G' as an induced subgraph. Next, it follows from **(i)** and the definition of A' that C is complete to U.

Let P be as in the definition of a quasi-homogeneous set decomposition and suppose that P is not perfect. Since P is an induced subgraph of G, it does not have an induced four-edge antipath or an induced five-edge path. It follows that P contains an induced cycle F of length five. Let $f_1, f_2, \dots, f_5$ be the vertices of F in order. Let P^* be obtained from P by deleting all edges between $U \cap V(P)$ and $N \cap V(P)$. It follows from (3.2.2) that P^* is perfect. Therefore, F is not an induced subgraph of P^*. It follows that there exist two vertices $a \in U$ and $b \in N$ that are adjacent in G, such that $a, b \in V(F)$, say $f_1 = a$ and $f_2 = b$.

It follows from **(ii)** that A is anticomplete to U. Hence, because f_1 and f_2 are adjacent, it follows that $f_2 \notin A$ and therefore $f_2 \in A'$. It follows from the definition of A' that $f_1 \notin V(H) \cup X' \cup Y' \cup Z$ and hence $f_1 \in X \cup Y$. Now let us consider f_3. Since f_3 is adjacent to f_2, it follows that $f_3 \in X \cup Y \cup A' \cup C$. If $f_3 \in A'$, then f_1-f_2-f_3 is an induced path with $f_1 \in X \cup Y$ and $f_2, f_3 \in X' \cup Y'$, contrary to **(iii)**. Since $f_1 \in X \cup Y$, C is complete to $X \cup Y$, and f_3 is nonadjacent to f_1, it follows that $f_3 \notin C$, and therefore $f_3 \in X \cup Y$. Now let us consider f_4 and f_5. If both f_4 and f_5 are in $X' \cup Y'$, then f_3-f_4-f_5 is an induced path with $f_3 \in X \cup Y$ and $f_4, f_5 \in X' \cup Y'$, contrary to **(iii)**. Therefore, from the symmetry, we may assume that $f_4 \notin X' \cup Y'$. Since f_4 is adjacent to f_3, this implies that $f_4 \in V(H) \cup C \cup X \cup Y \cup Z$. Since f_4 is not adjacent to f_1 and C is complete to f_1, it follows that $f_4 \notin C$. Therefore, **(i)** implies that f_4 is complete to C. This proves that C is complete to $\{f_1, f_3, f_4\}$.

Let K' be the component of A' that contains f_2. We claim that f_1 is complete to K'. For suppose not. Because f_1 is adjacent to $f_2 \in K'$, it follows that there exist adjacent $k_1, k_2 \in K'$ such that f_1 is adjacent to k_1 and not to k_2. But now f_1-k_1-k_2 is an induced path with $f_1 \in X \cup Y$ and $k_1, k_2 \in X' \cup Y'$, contrary to **(iii)**. This proves that f_1 is complete to K' and, from the symmetry, that f_3 is complete to K'. Similarly, and since $V(H)$ is anticomplete to K', it follows that f_4 is anticomplete to K'. Since K' is not complete to C by the definition of A', we may choose $f_2' \in K'$ and $c \in C$ such that f_2' is nonadjacent to c (perhaps by choosing $f_2' = f_2$). It follows from the previous that f_2' is adjacent to f_1 and f_3. Therefore, f_1-f_2'-f_3-f_4 is an induced path. It follows from the previous that c is complete to $\{f_1, f_3, f_4\}$ and nonadjacent to f_2, contrary to (3.4.1). This proves **(iv)**. □

In view of **(iv)**, we may from now on assume that no 6-gon in G has a (y)-vertex.

(v) *If $Z \neq \emptyset$, then the lemma holds.*

Suppose that $Z \neq \emptyset$. From the symmetry, we may assume that there exists $z \in Z$ such that z is adjacent to h_2 and h_6. Let Z_1' be the set of vertices in Z that are adjacent to h_2 and h_6 and let $Z_1 = Z_1' \cup \{h_1\}$. It follows from the definition of Z_1 that $|Z_1| \geq 2$. Let R be the set of vertices in $V(G) \setminus Z_1$ with a neighbor in Z_1 and let $S = V(G) \setminus (Z_1 \cup R)$. We claim that (Z_1, S, R) is a homogeneous set decomposition of G. For suppose not. Then there exist $v \in V(G) \setminus Z_1$ and $x, y \in Z_1$ such that v is adjacent to x and nonadjacent to y. It follows from the definition of Z_1 that $v \notin V(H)$. Let $H' = x$-h_2-h_3-$\dots$-h_6-x. Since H' has no (y)-vertex and C is complete to Z_1 by **(i)**, it follows from (3.5.1) that v is either an (x)-vertex or a (z)-vertex for H'. It follows that v is anticomplete to h_4 and, since $v \notin Z_1$, v is adjacent to at least one of h_3, h_5. From the symmetry, we may assume that v is adjacent to h_3. It follows from the fact that v is either an (x)-vertex of a (z)-vertex for H', that v is nonadjacent to h_6. Since y-h_6-x-v-h_3-h_4 is not an induced five-edge path, it follows that x is adjacent to y. If v is nonadjacent to h_2, then x-h_3-y-v-h_2 is an induced four-edge antipath, a contradiction. Thus, v is adjacent to h_2 and hence v is a (z)-vertex for H', and v is nonadjacent to h_5. Now, the adjacency of v with respect to the 6-gon y-h_2-h_3-$\dots$-h_6-y contradicts (3.5.1). This proves that (Z_1, R, S) is a homogeneous set decomposition, and hence a quasi-homogeneous set decomposition, of G . This proves **(v)**. □

In view of **(v)**, we may from now on assume that $Z = \emptyset$. Let X_1 and X_2 be the vertices in X that are complete to $\{h_1, h_3, h_5\}$ and $\{h_2, h_4, h_6\}$, respectively. Now, $(\{h_1, h_3, h_5\}, \{h_2, h_4, h_6\}, X_1, X_2, C, A \cup X' \cup X'')$ is a Σ-join. This proves (3.5.2). ■

We are now in a position to prove (3.0.15):

(3.0.15). $\nu(G) \leq 3$ *for every* $G \in \mathrm{Forb}(P_4, P_4^c)$.

Proof. We prove the theorem by induction on $|V(G)|$. Let $G \in \mathrm{Forb}(P_4^c, P_5)$. Suppose first that G contains a 6-gon with a center. Then it follows from (3.5.2) that G admits either a quasi-homogeneous set decomposition or a Σ-join. If G admits a quasi-homogeneous set decomposition, then it follows from (3.2.3) and the induction hypothesis that G is 3-narrow. Otherwise, if G admits a Σ-join, then it follows from (3.2.4) and the induction hypothesis that G is 3-narrow. So we may assume that G contains no 6-gon with a center. Now let $v \in V(G)$. Clearly, $G|N(v)$ does not have C_6 as an induced subgraph. Therefore, $G|N(v) \in \text{Forb}(P_4^c, P_5, C_6)$ and hence, by Theorem 3.4.13, $G|N(v)$ is 2-narrow. By the induction hypothesis, it follows that $G|M(v)$ is 3-narrow. Since this is true for every $v \in V(G)$, it follows from (3.2.1) that G is 3-narrow. This proves (3.0.15). ■

4

Line graphs with fractionally strongly perfect complements and their applications

In the current and the next chapter, we are interested in *fractionally co-strongly perfect graphs*:

Definition. *A graph G is* fractionally co-strongly perfect *if and only if, for every induced subgraph H of G, there exists a function $w : V(H) \to [0,1]$ such that*

$$\sum_{v \in S} w(v) = 1, \text{ for every maximal stable set } S \text{ of } H. \tag{4.1}$$

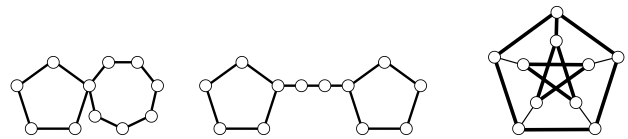
We call a function w that satisfies (4.1) *a* saturating vertex weighting *for H.*

The initial motivation for studying this graph property is its application in scheduling of communication in wireless networks. The purpose of the current chapter is two-fold. First, we want to show how fractional co-strong perfect graphs show up in a 'real-life' application in electrical engineering. Second, although the graph-theoretical proofs in this chapter are quite elementary, this chapter tries to convey the basic ideas that we use when we deal with 'strip-structures' in Chapter 5.

This chapter is an edited and shortened version of a paper **[7]**, in which we give a characterization of all graphs that satisfy the so-called Local Pooling conditions. Before we define what is meant by these conditions and describe how they relate to fractional co-strong perfection, we need a few definitions. First recall that line graphs are defined as follows.

Definition. *Let H be a graph. Let $L(H)$ be a graph whose vertex set is the edge set of H, and in which two vertices e_1, e_2 of $L(H)$ are adjacent if and only e_1 and e_2 share an endpoint in H. Then $L(H)$ is called the* line graph of *H. Every graph that is the line graph of some graph is called a* line graph.

The definition of a line graph implies that there is a one-to-one correspondence between the maximal stable sets of a line graph $L(H)$ and the maximal matchings of H. Similarly, there is a one-to-one



(a) $D_0^{5,7}$ (b) $D_3^{5,5}$ (c) Petersen graph

Figure 4.1: *Graphs (a) and (b): examples of graphs from the family $D_k^{p,q}$, all of which fail the LoP conditions. Graph (c): the Petersen graph. This graph does not satisfy the LoP conditions because it contains, among other graphs, C_6 and $D_1^{5,5}$ (bold edges) as subgraphs.*

correspondence between the induced subgraphs of $L(H)$ and (the line graphs of) the subgraphs of H. This implies the following equivalent definition of fractional co-strong perfection of a line graph $L(H)$ in terms of the graph H underlying it:

Definition. *For a graph H, the line graph $L(H)$ is* fractionally co-strongly perfect *if and only if, for every subgraph H' of H, there exists a function $w : E(H') \to [0,1]$ such that*

$$\sum_{e \in M} w(e) = 1, \text{ for every maximal matching } M \text{ of } H. \tag{4.2}$$

We call a function w that satisfies (4.2) a saturating edge weighting *for H'.*

This definition is exactly what is meant by the Local Pooling conditions: a graph H is said to *satisfy the Local Pooling (LoP) conditions* if $L(H)$ is fractionally co-strongly perfect.

Main results

Define the following families of graphs. For $k \geq 3$, let C_k be a cycle with k edges (or, equivalently, k vertices). For $k \geq 0$ and $p, q \in \{5, 7\}$, let $D_k^{p,q}$ be the graph formed by the union of two cycles of size p and q joined by a k-edge path (where $k \geq 0$). If $k = 0$, the cycles share a common vertex (see Fig. 4.1-(a) and 4.1-(b)). Let $\mathcal{C} = \{C_k \mid k \geq 6, k \neq 7\} \cup \{D_k^{p,q} \mid k \geq 0; p, q \in \{5, 7\}\}$. For two graphs G and H, we say that G *contains H as a subgraph* if G has a subgraph that is isomorphic to H. We will say that a graph G is *$\mathcal{C}$-free*, if it does not contain any graph $F \in \mathcal{C}$ as a subgraph. It is easy to see that a graph satisfies the LoP conditions if and only if all its connected components satisfy the LoP conditions. So we may assume without loss of generality that all graphs in this chapter are connected.

The results in chapter consist of two parts. First, we give a structural description of all $\mathcal{C}$-free graphs and we will use this description to prove the following theorem:

Theorem 4.0.3. *A graph G satisfies the LoP conditions if and only if G is $\mathcal{C}$-free.*

Theorem 4.0.3 shows that if a graph G does not satisfy the LoP conditions, then G contains some $F \in \mathcal{C}$ as a subgraph. For example, it was previously shown that the Petersen graph (Fig. 4.1-(c)) fails the LoP conditions **[36]**. Using Theorem 4.0.3 we can immediately see this from the fact that it contains, for example, C_6 and $D_1^{5,5}$ as a subgraph.

Testing whether a graph satisfies the LoP conditions previously required enumerating all maximal matchings (of which there are an exponential number) and solving a Linear Program, and repeating this for every subgraph **[23]**. The weakness of this approach is its large computational effort. In Section 4.3, we present the third result, which uses the structure of $\mathcal{C}$-free graphs to construct an algorithm that decides in linear time whether a graph satisfies the LoP conditions, as described in the following theorem:

Theorem 4.0.4. *It can be decided in $O(|V(G)|)$ time whether a graph G satisfies the LoP conditions.*

We remark that for a given line graph G, the graph H such that $G = L(H)$ can be constructed in linear time **[49]** and, therefore, Theorem 4.0.4 implies that it can be determined in linear time whether a given line graph is fractionally co-strongly perfect.

Organization of this chapter

This chapter is organized as follows. In Section 4.1, we will give some background that forms the motivation for studying fractionally co-strongly perfect graphs. In Section 4.2, we give a characterization of all graphs that satisfy the Local Pooling conditions, thus giving a description of all line graphs that are fractionally co-strongly perfect. Finally, in Section 4.3, we give a linear-time algorithm for deciding whether a given graph satisfies the Local Pooling conditions.

4.1 Background: communication in wireless networks

Communication under primary interference

Consider a wireless communication network that consists of transmitters and receivers that are connected by wireless connections. Define the graph $G = (V, E)$, where $V = \{1, \ldots, n\}$ is the set of transmitters and receivers, and $E \subseteq \{ij : i, j \in V, i \neq j\}$ is a set of links indicating pairs of vertices between which data flow can occur. We refer to this graph as the *network graph* of the wireless communication network. We assume that every vertex can transmit as well as receive. Following the model of **[10, 23, 36, 53]**, assume that time is slotted and that packets are of equal size, each packet requiring one time slot of service to be transmitted across a link. The model considers only single-hop traffic, which means that every packet that arrives at a vertex i has a known destination j such that $ij \in E$. Thus, we do not take into account the usual routing problem in which it has to be decided which route packets should take in order to be transmitted from a source i to a nonadjacent destination j. A queue is associated with each edge in the graph. We assume that the stochastic arrivals to edge ij have long term rates λ_{ij} and are independent of each other. We denote by $\vec{\lambda}$ the vector of the arrival rates λ_{ij} for every edge ij. For more details regarding the queue evolution process under this model, see **[10, 23, 36]**.

For a graph G, let $\mathbf{M}(G)$ be a 0-1 matrix with $|E|$ rows, whose columns represent the maximal matchings of G. A *scheduling algorithm* selects a set of edges to activate at each time slot and transmits packets on those edges. However, they have to be chosen in such a way that the links that are activated simultaneously do not interfere with each other. The interference model that we adopt here is the *primary interference* model, in which two links interfere with each other if and only if they share an endpoint. Thus, since the activated links must not interfere under primary interference constraints, the selected edges form a matching. In other words, the scheduling algorithm picks a column $\pi(t)$ from the maximal matching matrix $\mathbf{M}(G)$ at every time slot t. If $\pi_k(t) = 1$, one of the two vertices along edge e_k can transmit, and the associated queue is decreased by one. We define the *stability region* (or *capacity region*) of a graph as follows.

Definition. [53] *The* stability region *of a graph G is defined by*

$$\Lambda^* = \left\{ \vec{\lambda} \,|\vec{\lambda} < \vec{u} \;\; \textit{for some } \vec{u} \in Co(\mathbf{M}(G)), \right\},$$

where $Co(\mathbf{M}(G))$ is the convex hull of the columns of $\mathbf{M}(G)$ and the inequality operator is taken element-wise.

Fix a (network) graph G. A scheduling algorithm is *stable for* $\vec{\lambda}$ if the Markov chain that represents the evolution of the queues (under this algorithm and the given arrival rates) is positive recurrent[1]. A scheduling algorithm is *stable* if it is stable for all arrivals $\vec{\lambda} \in \Lambda^*$. It is known that, if $\vec{\lambda} \in \mathbb{R}^n_+ \setminus \Lambda^*$, then there exists no stable algorithm for $\vec{\lambda}$.

The *efficiency ratio* γ^* of a given algorithm is the largest value γ such that the algorithm is stable for all $\vec{\lambda} \in \gamma\Lambda^*$. In simple words, γ^* is the fraction of the stability region for which the queues are bounded. If an algorithm has an efficiency ratio γ^*, then we say that this algorithm *achieves* $100\gamma^*\%$ *throughput*. Thus, a stable scheduling algorithm achieves 100% throughput by definition. We therefore also refer to such an algorithm as *throughput-optimal*. It was shown in **[53]** that the Maximum Weight Matching algorithm that selects a matching with maximum total queue size at each slot is stable. The results of **[53]** have been extended to various settings of wireless networks and input-queued switches (see *e.g.* **[2, 30, 45]**). However, algorithms based on **[53]** require the repeated solution of a global optimization problem, taking into account the queue backlog of every link. For example, even under simple primary interference constraints, a maximum weight matching problem has to be solved in every slot, requiring an $O(n^3)$ algorithm.

Hence, there has been an increasing interest in simple (potentially distributed) algorithms. One such algorithm is the *Greedy Maximal Scheduling* (GMS) algorithm (also termed Maximal Weight Scheduling or Longest Queue First - LQF). This algorithm selects the set of served links greedily according to the queue lengths **[34, 42]**. Namely, at each step, the algorithm selects the heaviest link (*i.e.* with longest queue size), and removes it and the links with which it interferes from the list of candidate links. The algorithm terminates when there are no more candidate links. Such an algorithm can be implemented in a distributed manner **[34, 40, 41]**. It was shown that the GMS algorithm achieves

[1]A (discrete) Markov chain $\{X_n\}_{n=0}^{\infty}$ is said to be *positive recurrent* if all of its states are positive recurrent, *i.e.* $\mathbb{E}(\inf\{n \geq 1 : X_n = i\} \mid X_0 = i) < \infty$ for all states i. In our application, the Markov chain being positive recurrent means that the queue lengths do not tend to infinity as $n \to \infty$.

50% throughput in general graphs under primary interference constraints **[42]**.

Although in arbitrary graphs the worst case performance of GMS can be very low, there are some graphs in which 100% *throughput is achieved*. Particularly, Dimakis and Walrand **[23]** presented sufficient conditions for GMS to provide 100% throughput. These conditions are referred to as *Local Pooling* (LoP) and are related to the structure of the graph. Based on these conditions, it was shown that GMS achieves maximum throughput in trees under primary interference **[37, 61]**, in $2 \times n$ bipartite graphs (which model $2 \times n$ switches) **[10]**. We briefly reproduce the definitions of Local Pooling (LoP) presented in **[10, 23]**.[2]

Definition. *A graph G satisfies* Subgraph Local Pooling (SLoP)*, if there exists $\alpha \in [0, 1]^{|E|}$ such that $\alpha^T \mathbf{M}(G) = \mathbf{e}^T$, where* **e** *denotes the vector having each entry equal to one. A graph G satisfies the* Local Pooling (LoP) *conditions, if every subgraph G' of G satisfies Subgraph Local Pooling.*

We notice that this definition is equivalent to saying that $L(G)$ is fractionally co-strongly perfect. The Local Pooling conditions are important because of the following theorem:

Theorem. **[23]** *If a graph satisfies the LoP conditions, then GMS achieves 100% throughput.*

In other words, our result describes the graphs for which GMS achieves full throughput.

General interference; interference graphs

This chapter only deals with primary interference. However, it is possible to generalize this model by introducing interference graphs. Based on the network graph and the interference constraints, the interference between network links can be modeled by an *interference graph* (or a *conflict graph*) $G_I = (V_I, E_I)$ **[35]**. We assign $V_I = E$. Thus, each edge e_k in the network graph is represented by a vertex v_k in the interference graph, and an edge $v_i v_j$ in the interference graph indicates a conflict between network graph links e_i and e_j (*i.e.* transmissions on e_i and e_j cannot take place simultaneously). Under primary interference, the interference graph G_I corresponds to the *line graph* of G.

An example of a more general interference model is the *k-hop interference model*, which states that two links e_1, e_2 interfere with each other if and only if there exists a path with at most k vertices between one of the endpoints of e_1 and one of the endpoints of e_2. Thus, primary interference is a equivalent to 1-hop interference.

The model and the LoP theory described so far extend to interference graphs. The vertices of G_I correspond to queues to which packets arrive according to a stochastic process at every time slot. A scheduling algorithm must pick an stable set at each slot so that neighboring vertices will not be activated simultaneously. Each column of the matrix $\mathbf{M}(G_I)$ corresponds to a maximal stable set of G_I. An algorithm which selects the stable set with the largest weights, *i.e.* which solves the (NP-hard) Maximum Weight Stable Set Problem, is stable. The corresponding SLoP condition is that there exists

[2] This definition slightly differs from that in **[10]** by setting the sum equal to $\mathbf{e}^T$ instead of $c\mathbf{e}^T$, where c is a positive constant.

a vector $\alpha \in [0,1]^{|V_I|}$ that assigns a weight $\alpha(u)$ to each vertex u such that $\sum_{u \in I} \alpha(u) = 1$ for every maximal stable set I in G_I. Similarly, the corresponding LoP condition is that SLoP is satisfied by all *induced* subgraphs. Therefore, the LoP condition for interference graphs is equivalent to requiring that the interference graph is fractionally co-strongly perfect. There are a few families of graphs for which it is known that the interference graph version of the LoP condition holds **[61]**, but there is no exact description of such graphs.

Contributions in terms of wireless network communication

While it is known that under primary interference some graph families (mainly trees and $2 \times n$ bipartite graphs) satisfy the LoP conditions, the exact structure of graphs that satisfy the LoP conditions was not characterized. In this chapter, we use graph theoretic methods to obtain the structure of all the graphs that satisfy the LoP conditions (in these graphs GMS achieves 100% throughput). This allows us to develop an algorithm that checks if a graph satisfies the LoP conditions in time linear in the number of vertices, significantly improving over any other known method. We note that although primary interference constraints may not hold in many wireless networking technologies, the characterization provides an important theoretical understanding regarding the performance of simple greedy algorithms. It also shows that the $2 \times n$ switch is the largest switch for which 100% throughput is guaranteed.

From a practical point of view, identifying graphs that satisfy the LoP conditions can provide important building blocks for partitioning a graph (*e.g.* via channel allocation) into subgraphs in which GMS performs well **[10]**. Another possible application is to add artificial interference constraints to a graph that does not satisfy the LoP conditions in order to turn it into a LoP-satisfying graph. Adding such constraints may decrease the stability region but would enable GMS to achieve a large portion of the new stability region.

4.2 Graphs that satisfy the local pooling conditions

The goal of this section is to prove Theorem 4.0.3. We start with a structural description of $\mathcal{C}$-free graphs, and then use it to prove Theorem 4.0.3.

4.2.1 The structure of $\mathcal{C}$-free graphs

The reason for our interest in $\mathcal{C}$-free graphs is the fact (which will be proved in Subsection 4.2.2) that the class of $\mathcal{C}$-free graphs is precisely the class of graphs that satisfy the LoP conditions.

We will describe the structure of $\mathcal{C}$-free graphs in terms of the so-called 'block decomposition'. Let G be a connected graph. We call $x \in V(G)$ a *cut-vertex of* G, if $G - x$ is not connected. We call a maximal connected induced subgraph B of G such that B has no cut-vertex a *block of* G. Let $B_1, B_2, \dots, B_q$ be the blocks of G. We call the collection $\{B_1, B_2, \dots, B_q\}$ the *block decomposition* *of* G. It is known that the block decomposition is unique and that $E(B_1), E(B_2), \dots, E(B_q)$ forms a partition of $E(G)$ (see *e.g.* **[58]**). Furthermore, the vertex sets of every two blocks intersect in at most one vertex and this vertex is a cut-vertex of G.

Block decompositions give a tree-like decomposition of a graph in the following sense. Construct the *block-cutpoint graph of* G by keeping the cut-vertices of G and replacing each block B_i of G by a vertex b_i. Make each cut-vertex v adjacent to b_i if and only if $v \in V(B_i)$. It is known that the block-cutpoint graph of G forms a tree (see *e.g.* **[58]**). With this tree-like structure in mind, we say that a block B_i is a *leaf block* if it contains at most one cut-vertex of G. Clearly, if $q \geq 2$, then $\{B_i\}_{i=1}^{q}$ contains at least two leaf blocks.

It turns out that the block decomposition of an $\mathcal{C}$-free graph is relatively simple in the sense that there are only two types of blocks. The types are defined by the following two families of graphs. Examples of these families appear in Fig. 4.2.

$\mathcal{B}_1$: Construct $\mathcal{B}_1$ as follows. Let H be a graph with $V(H) = \{c_1, c_2, \dots, c_k\}$, with $k \in \{5, 7\}$, such that

1. c_1-c_2-$\cdots$-c_k-c_1 is a cycle;
2. if $k = 5$, then the other adjacencies are arbitrary; if $k = 7$, then all other pairs are nonadjacent, except possibly $\{c_1, c_4\}$, $\{c_1, c_5\}$ and $\{c_4, c_7\}$.

Then, $H \in \mathcal{B}_1$.
Now iteratively perform the following operation. Let $H' \in \mathcal{B}_1$ and let $x \in V(H')$ with $\deg(x) = 2$. Construct H'' from H' by adding a vertex x' such that $N(x') = N(x)$. Then, $H'' \in \mathcal{B}_1$. We say that a graph is *of the* $\mathcal{B}_1$ *type* if it is isomorphic to a graph in $\mathcal{B}_1$.

$\mathcal{B}_2$: Let $\mathcal{B}_2 = \{K_2, K_3, K_4\} \cup \{K_{2,t}, K_{2,t}^+ \mid t \geq 2\}$, where $K_{2,t}^+$ is constructed from $K_{2,t}$ by adding an edge between the two vertices on the side that has cardinality 2. We say that a graph is *of the* $\mathcal{B}_2$ *type*, if it is isomorphic to a graph in $\mathcal{B}_2$.

In simple words, graphs of the $\mathcal{B}_1$ type are constructed by starting with a cycle of length five or seven. Then we may add some additional edges between vertices of the cycle, subject to some constraints. Finally, we may iteratively take a vertex x of degree 2 and add a clone x' of x. It will turn out that $\mathcal{C}$-free graphs have at most one block of the $\mathcal{B}_1$ type and that all other blocks are of the $\mathcal{B}_2$ type. This means that $\mathcal{C}$-free graphs can be constructed by starting with a block that is either of the $\mathcal{B}_1$ or of the $\mathcal{B}_2$ type, and then iteratively adding a block of the $\mathcal{B}_2$ type by 'gluing' it on an arbitrary vertex.

Fig. 4.2 shows an example of an $\mathcal{C}$-free graph. The tree-like structure is clearly visible. The graph has one block of the $\mathcal{B}_1$ type with $k = 7$. This block consists of a cycle of length 7 together with two clones. The other blocks are of the $\mathcal{B}_2$ type. Some of them are attached to the block of the $\mathcal{B}_1$ type through a cut-vertex. Others are attached to other blocks of the $\mathcal{B}_1$ type. Notice that trees and $2 \times n$ complete bipartite graphs, which were previously known to satisfy the LoP conditions **[37, 10]**, are, as should be expected, subsumed by this structure.

The goal of this subsection is to prove the following formal version of the characterization given above:

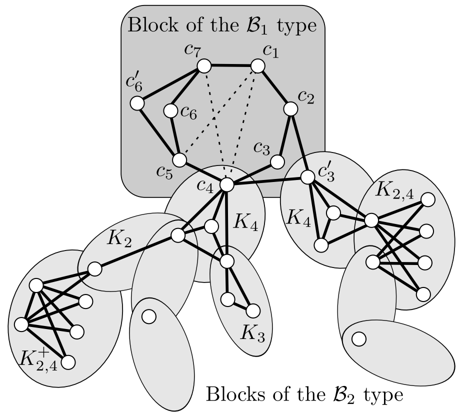


Figure 4.2: *An example of an $\mathcal{C}$-free graph (the dashed edges may or may not be present). The ellipses show the blocks of the graph.*

Theorem 4.2.1. *Let G be a connected graph and let $\{B_1, B_2, \dots, B_q\}$ be the block decomposition of G. Then G is $\mathcal{C}$-free if and only if there is at most one block that is of the $\mathcal{B}_1$ type and all other blocks are of the $\mathcal{B}_2$ type.*

The proof of the 'if' direction is straightforward. Here, we will give a proof of the 'only-if' direction in a number of steps. For a block B in an $\mathcal{C}$-free graph, its type depends on the size of the longest cycle in B. It will turn out that if B contains a cycle of length 5 or 7, then B is of the $\mathcal{B}_1$ type. Otherwise, B is of the $\mathcal{B}_2$ type. We have the following result on blocks that have a cycle of length five or seven.

(4.2.2) *Let G be an $\mathcal{C}$-free graph and let B be a block of G. Let F be a cycle in B that has maximum length. If $|V(F)| \geq 5$, then B is of the $\mathcal{B}_1$ type.*

Next, we deal with blocks that do not contain a cycle of length 5 or 7. It follows from the definition of $\mathcal{C}$-free graphs that such blocks do not have cycles of length at least 5. Maffray **[44]** proved the following theorem:

Theorem 4.2.3. [44] *Let G be a graph. Then, the following statements are equivalent:*

(1) *G does not contain any odd cycle of length at least 5.*

(2) *For every connected subgraph G' of G, either G' is isomorphic to K_4, or G' is a bipartite graph, or G' is isomorphic to $K_{2,t}^+$ for some $t \geq 1$, or G' has a cut-vertex.*

Theorem 4.2.3 implies the following lemma.

(4.2.4) *Let G be an $\mathcal{C}$-free graph and let B be a block of G. Suppose that B contains no cycle of length at least 5. Then, B is of the $\mathcal{B}_2$ type.*

Proof. Since B has no cycle of length at least 5 and B has no cut-vertex, it follows from Theorem 4.2.3 that either B is a bipartite graph, or B is isomorphic to K_4, or B isomorphic to $K_{2,t}^+$. In the latter two cases, we are done. So suppose that B is a bipartite graph. Let $V(G) = X \cup Y$ such that X and Y are stable sets. If $|X| \leq 1$, then $x \in X$ is a cut-vertex, a contradiction. From the symmetry, it follows that $|X| \geq 2$ and $|Y| \geq 2$. Now suppose $x \in X$ is nonadjacent to $y \in Y$. Since B is 2-connected, it follows that there are two edge-disjoint paths P_1 and P_2 from x to y. Since x and y are nonadjacent and B is bipartite, it follows that $|E(P_1)| \geq 3$ and $|E(P_2)| \geq 3$. But now x-P_1-y-P_2-x is a cycle of length at least six, a contradiction. It follows that X is complete to Y. If $|X| \geq 3$ and $|Y| \geq 3$, then clearly, B contains a cycle of length six, a contradiction. Therefore, at least one of X, Y has size exactly 2. Hence, B is isomorphic to $K_{2,t}$ with $t = \max\{|X|, |Y|\}$ and therefore B is of the $\mathcal{B}_2$ type. This proves (4.2.4). ■

We are now ready to prove Theorem 4.2.1, the statement of which we repeat for clarity:

Theorem 4.2.1. *Let G be a connected graph and let $\{B_1, B_2, \dots, B_q\}$ be the block decomposition of G. Then G is $\mathcal{C}$-free if and only if there is at most one block that is of the $\mathcal{B}_1$ type and all other blocks are of the $\mathcal{B}_2$ type.*

Proof. Let G be an $\mathcal{C}$-free graph and let $\{B_1, B_2, \dots, B_m\}$ be the block decomposition of G. For every $i \in [m]$, if B_i contains a cycle of length 5 or 7, it follows from (4.2.2) that B_i is of the $\mathcal{B}_1$ type. Otherwise, it follows from from (4.2.4) that B_i is of the $\mathcal{B}_2$ type. Now suppose that there are $i \neq j$ and $p, q \in \{5, 7\}$ such that B_i contains a cycle T_1 of length p and B_j contains a cycle T_2 of length q. Since G is connected, there exists a path P of length $k \geq 0$ from a vertex in T_1 to a vertex in T_2. Since T_1 and T_2 are subgraphs of different blocks, T_1 and T_2 share at most one vertex. If they share a vertex, then $k = 0$. Now the edges of T_1, T_2, P form a graph isomorphic to $D_k^{p,q}$, a contradiction. This proves Theorem 4.2.1. ■

4.2.2 A graphs G satisfies the local pooling conditions if and only if G is $\mathcal{C}$-free

Now that we have described the structure of all $\mathcal{C}$-free graphs, we use this structure to prove Theorem 4.0.3 which states that a graph satisfies the LoP conditions, if and only if it is $\mathcal{C}$-free. It was shown in **[10]** (Theorems 2 and 3) that all cycles of length $k \geq 6, k \neq 7$ fail SLoP.[3] Therefore, such cycles do not appear as subgraphs in graphs that satisfy the LoP conditions. Before we prove that the same is true for the graphs $D_k^{p,q}$, we will need a lemma:

(4.2.5) *Let $m \in \{5, 7\}$ and let $q \geq 0$. Let G' be a graph and let F be a m-cycle disjoint from G'. Let $v \in V(G')$ such that there exists a matching in G' that covers all neighbors of v in G', but not v itself. Let G be the graph constructed from the disjoint union of G' and F by adding a path P of length q between $f \in V(F)$ and v. Then every saturating edge weighting α for G satisfies $\alpha(e) = 0$ for every $e \in E(F) \cup E(P)$.*

[3]Although the case considered in **[10]** pertains to interference graphs, the network graph case is identical since the interference graph (under primary interference) of a cycle is a cycle of the same length.

Proof. Let $f_1, f_2, \dots, f_m$ be the vertices of F in order and let $p_1, p_2, \dots, p_q$ be the vertices of P. We may assume that $f = f_m$, $p_1 = f$ and $p_q = v$. We use induction on q. First suppose that $q = 0$, *i.e.* $v = f_m$. We will prove this for the case when $m = 5$. The case when $m = 7$ is analogous. Let M be a maximal matching in G' that covers v. Let $M_1 = M \cup \{f_1f_2, f_3f_4\}$ and let $M_2 = M \cup \{f_2f_3\}$. Since α is a saturating edge weighting and M_1 and M_2 are maximal matchings, it follows that $\alpha(f_2f_3) = \alpha(f_1f_2) + \alpha(f_3f_4)$. Now let M' be a maximal matching in G' that does not cover v. Let $M'_1 = M' \cup \{f_1v, f_2f_3\}$ and $M'_2 = M' \cup \{f_1v, f_3f_4\}$. Since α is a saturating edge weighting and M'_1 and M'_2 are maximal matchings, it follows that $\alpha(f_2f_3) + \alpha(f_1v) = \alpha(f_3f_4) + \alpha(f_1v)$. Hence, $\alpha(f_2f_3) = \alpha(f_3f_4)$. Using the symmetry, it follows that $\alpha(f_2f_3) = \alpha(f_1f_2)$. Combining this with the equality found above, it follows that $\alpha(f_2f_3) = 2\alpha(f_2f_3)$ and hence that $\alpha(f_1f_2) = \alpha(f_2f_3) = \alpha(f_3f_4) = 0$. Finally, let M'' be a maximal matching in G' that covers all neighbors of v but not v itself. Let $M''_1 = M'' \cup \{f_1v, f_2f_3\}$ and $M''_2 = M'' \cup \{f_1f_2, f_3f_4\}$. Since α is a saturating edge weighting and M''_1 and M''_2 are maximal matchings, it follows that $\alpha(f_1v) + \alpha(f_2f_3) = \alpha(f_1f_2) + \alpha(f_3f_4) = 0$. Hence, $\alpha(f_1v) = 0$ and, from the symmetry, $\alpha(f_4v) = 0$. This proves the claim for $q = 0$.

Next, suppose that $q \geq 1$. It follows from the induction hypothesis that $\alpha(e) = 0$ for all $e \in (E(F) \cup E(P)) \setminus \{p_{q-1}p_q\}$. Let M be a matching in G' that covers all neighbors of v but not v itself. Let M_1 be a maximal matching in $G|(V(F) \cup V(P))$ that covers v and let M_2 be a maximal matching in $G \setminus (V(F) \cup V(P))$ that does not cover v. Since $M \cup M_1$ and $M \cup M_2$ are maximal matchings, it follows that $\alpha(M_1) = \alpha(M_2)$. Since $\alpha(M_2) = 0$, it follows that $\alpha(p_{q-1}p_q) = 0$. This proves (4.2.5). ■

This allows us to prove the following:

(4.2.6) *$D_k^{p,q}$ fails SLoP for all $p, q \in \{5, 7\}$, $k \geq 0$.*

Proof. Let $k \geq 0$, $p, q \in \{5, 7\}$ and suppose that $D_k^{p,q}$ satisfies SLoP. Then there exists a saturating edge weighting α for $D_k^{p,q}$. It follows from (4.2.5) applied to $D_k^{p,q}$ that $\alpha(e) = 0$ for all $e \in E(D_k^{p,q})$. This is clearly not a saturating edge weighting for $D_k^{p,q}$, a contradiction. This proves (4.2.6). ■

The results from **[10]** together with (4.2.6) imply the following result:

(4.2.7) *Graphs that satisfy the LoP conditions are $\mathcal{C}$-free.*

Proof. Let G be a graph that satisfies the LoP conditions. By the definition of the LoP conditions, every subgraph H of G satisfies SLoP. Since every graph in $\mathcal{C}$ fails SLoP, it follows that G does not contain any graph in $\mathcal{C}$ as a subgraph. This proves (4.2.7). ■

(4.2.7) settles the 'only-if' direction of Theorem 4.0.3. To prove the 'if' direction, we will start with a useful lemma:

(4.2.8) *Let G be a graph and $x, x' \in V(G)$ such that* $\deg(x) = 2$ *and x' is a clone of x. Then, G satisfies SLoP.*

Proof. Let x and x' be as in the claim and let $\{z_1, z_2\} = N(x)$. Define $\alpha \in [0, 1]^{|E|}$ by

$$\alpha(e) = \begin{cases} 1/2 & \text{if } e \text{ is incident with } z_1 \text{ or } z_2, \text{ and } e \neq z_1 z_2 \\ 1 & \text{if } e = z_1 z_2 \\ 0 & \text{otherwise.} \end{cases}$$

To see that α is a saturating edge weighting for G, let M be a maximal matching in G'. If $z_1 z_2 \in M$, then no other edge in M is incident with z_1 or z_2 and hence $\sum_{e \in M} \alpha(e) = 1$. Therefore we may assume that $z_1 z_2 \notin M$. It suffices to show that M covers both z_1 and z_2. So let us assume to the contrary that M does not cover z_1. Since M is a matching, at most one of $xz_2, x'z_2$ is in M. From the symmetry, we may assume that $xz_2 \notin M$. But now we may add xz_1 to the matching and obtain a larger matching, contrary to the maximality of M. This proves (4.2.8). ■

The following lemma is the crucial step in settling the 'if' direction of Theorem 4.0.3.

(4.2.9) *Every connected $\mathcal{C}$-free graph satisfies SLoP.*

Proof. The proof is by induction on $|E(G)|$. Let $\{B_1, B_2, \dots, B_q\}$ be the block decomposition of G. It follows from Theorem 4.2.1 that B_i is either of the $\mathcal{B}_1$ type or of the $\mathcal{B}_2$ type, and for at most one value of i, B_i is of the $\mathcal{B}_1$ type. Since, inductively, every proper subgraph of G satisfies SLoP, it suffices to find a saturating edge weighting α for G.

Suppose first that G has a leaf block B_i of the $\mathcal{B}_2$ type. If $q = 2$, then let x be the cut-vertex of G in $V(B_i)$. If $q = 1$, let $x \in V(B_i)$ be arbitrary. There are four cases:

(1) B_i is isomorphic to K_2: let x, v denote the vertices of B_i. Let $\alpha(e) = 1$ for all edges incident with x and $\alpha(e) = 0$ for every other edge e. Let M be a maximal matching in G. If $xv \in M$, then, since M is a matching, M does not contain any other edge e with $\alpha(e) = 1$ and, hence, $\sum_{e \in M} \alpha(e) = 1$. If $xv \notin M$, then, since M is maximal, M contains an edge incident with x and, hence, $\sum_{e \in M} \alpha(e) = 1$. Since this is true for every maximal matching M of G, it follows that α is a saturating edge weighting for G.

(2) B_i is isomorphic to K_3: let x, v_1, v_2 denote the vertices of B_i and let $\alpha(e) = 1$ for all $e \in E(B_i)$ and $\alpha(e) = 0$ for every other edge e. Let M be a maximal matching in G. If $v_1 v_2 \in M$, then, since M is a matching, M does not contain either of xv_1, xv_2 and, hence, $\sum_{e \in M} \alpha(e) = 1$. If $v_1 v_2 \notin M$, then, since M is maximal and M is a matching, exactly one of xv_1, xv_2 is in M and, hence, $\sum_{e \in M} \alpha(e) = 1$. Since this is true for every maximal matching M of G, it follows that α is a saturating edge weighting for G.

(3) B_i is isomorphic to K_4: let x, v_1, v_2, v_3 denote the vertices of B_i and let $\alpha(v_1 v_2) = \alpha(v_1 v_3) = \alpha(v_2 v_3) = 1$ and $\alpha(e) = 0$ for all $e \in (E(G) \setminus \{v_1 v_2, v_1 v_3, v_2 v_3\})$. To see that this is a saturating edge weighting, let M be a maximal matching in G. Suppose that M does not contain any of the edges $v_1 v_2, v_1 v_3, v_2 v_3$. Since M does not contain $v_1 v_3$ and M is maximal, it follows that M contains either xv_1 or xv_3. Assume without loss of generality that $xv_1 \in M$. Now we may add

v_2v_3 to M to obtain a larger matching, a contradiction. It follows that $\sum_{e \in M} \alpha(e) = 1$. Since this is true for every maximal matching M of G, it follows that α is a saturating edge weighting for G.

(4) B_i is isomorphic to $K_{2,t}$ or $K_{2,t}^+$ for some $t \geq 2$: let $V(B_i) = V_1 \cup V_2$ such that $|V_1| = 2$ and V_2 is an stable set. Let $V_1 = \{y_1, y_2\}$ and let $V_2 = \{z_1, z_2, ..., z_t\}$.

First suppose that B_i is isomorphic to $K_{2,2}^+$ and $x \in V_2$. We may assume that $x = z_1$. Set $\alpha(y_1z_2) = \alpha(y_2z_2) = \alpha(y_1y_2) = 1$ and $\alpha(e) = 0$ for all other edges e. Let M be a maximal matching in G. Suppose that M does not use any of the edges y_1z_2, y_2z_2, y_1y_2. Since M is a matching, at least one of the edges xy_1, xy_2 is not in M, say xy_1. But now we may add y_1z_2 to M to obtain a larger matching, a contradiction. It follows that $\sum_{e \in M} \alpha(e) = 1$. Since this is true for every maximal matching M of G, it follows that α is a saturating edge weighting for G. This solves the case when B_i is isomorphic to $K_{2,2}^+$ and $x \in V_2$. So we may assume this is not the case.

We claim that B_i contains two vertices p, p' of degree 2 such that p' is a clone of p. Suppose that $x \in V_1$. Then let $p = z_1$, $p' = z_2$. It follows that $\deg(p) = \deg(p') = 2$ and p' is a clone of p. Therefore, we may assume that $x \in V_2$. We may assume that $x = z_1$. Suppose that $|V_2| \geq 3$. Then let $p = z_2$, $p' = z_3$. It follows that $\deg(p) = \deg(p') = 2$ and p' is a clone of p. So we may assume that $|V_2| = 2$. From the above, it follows that B_i is isomorphic to $K_{2,2}$. Let $p = y_1$, $p' = y_2$. It follows that $\deg(p) = \deg(p') = 2$ and p' is a clone of p.

Now the result follows from (4.2.8).

Thus, we may assume that G does not have a leaf block of the $\mathcal{B}_2$ type. Since if $q \geq 2$, G has at least two leaf blocks, and hence at least one leaf block of the $\mathcal{B}_2$ type, we may assume that $q = 1$ and $G = B_1$ is of the $\mathcal{B}_1$ type. First suppose that $V(G) \setminus V(C) \neq \emptyset$. Then it follows from the definition of $\mathcal{B}_1$ that there exist two vertices x, x' such that $\deg(x) = \deg(x') = 2$ and $N(x) = N(x')$. It follows from (4.2.8) that there exists a saturating edge weighting for G. So we may assume that $V(G) = V(C)$. Suppose first that $k = 5$. Recall that it follows from the definition of $\mathcal{B}_1$ that G is a 5-cycle plus some arbitrary additional edges. Clearly, no maximal matching has size 1. Hence, since $|V(G)| = 5$, it follows that every maximal matching in G has size exactly 2. Therefore, $\alpha(e) = 1/2$ for all $e \in E(G)$ is a saturating edge weighting for G. So we may assume that $k = 7$. Clearly, G has no maximal matching of size 1. It is also easy to see that G has no maximal matching of size 2. Hence, since $|V(G)| = 7$, it follows that every maximal matching in G has size exactly 3 and therefore $\alpha(e) = 1/3$ for all $e \in E(G)$ is a saturating edge weighting for G. This proves Theorem 4.0.3. ∎

We are now in a position to prove Theorem 4.0.3:

Proof of Theorem 4.0.3. (4.2.7) is the 'only-if' part of the theorem. For the 'if' part, since every subgraph of G is $\mathcal{C}$-free, it follows from (4.2.9) that every subgraph of G satisfies SLoP. Therefore, G satisfies the LoP conditions. This proves Theorem 4.0.3. ∎

4.3 Recognizing graphs that satisfy the LoP conditions

Having described the structure of graphs that satisfy the LoP conditions, we provide an efficient algorithm for testing whether a given graph satisfies the LoP conditions. The following is a useful observation:

(4.3.1) $|E(G)| \leq 2|V(G)|$ *for every* $\mathcal{C}$*-free graph* G.

Proof. We may assume that G is connected, because if not, then the lemma follows from considering each connected component of G. We first claim that $|E(B)| \leq 2|V(B)|$ for all $B \in \mathcal{B}_1$. Let $B \in \mathcal{B}_1$ and let C be a longest cycle in B. It follows from the definition of $\mathcal{B}_1$ that $|V(C)| \in \{5, 7\}$. Clearly, we have $|E(B_i)| \leq |V(C)| + 5 + 2(|V(B_i) \setminus V(C)|) \leq 2|V(C)| + 2(|V(B_i)| - |V(C)|) = 2|V(B_i)|$. This proves the claim. Next we claim that $|E(B)| \leq 2|V(B)| - 2$ for all $B \in \mathcal{B}_2$. If B is isomorphic to K_4, then $|E(B)| = 6 = 2|V(B)| - 2$. If B is isomorphic to $K_{2,t}$ or $K^+_{2,t}$ for some $t \geq 1$, then $|E(B)| \leq 1 + 2(|V(B)| - 2) < 2|V(B)| - 2$. This proves the claim.

Now let G be an $\mathcal{C}$-free graph and let $\{B_1, B_2, \dots, B_q\}$ be the block decomposition of G. We prove by induction on q that $|E(G)| \leq 2|V(G)|$. If $q = 1$, it follows immediately from the above that $|E(G)| = |E(B_1)| \leq 2|V(B_1)| = 2|V(G)|$. Next, let $q \geq 2$. Since G has at least two leaf blocks and at most one block is in $\mathcal{B}_1$, we may choose i such that B_i is a leaf block and B_i is of the $\mathcal{B}_2$ type. Let x be the unique cut-vertex of G that lies in B_i. By induction, the graph $G|(V(B_i) \setminus \{x\})$ has at most $2(|V(G)| - |V(B_i)| + 1)$ edges. From the above, since B_i is of the $\mathcal{B}_2$ type, it follows that $|E(B_i)| \leq 2|V(B_i)| - 2$. Hence, we have $|E(G)| \leq 2(|V(G)| - |V(B_i)| + 1) + 2|V(B_i)| - 2 = 2|V(G)|$. This proves (4.3.1). ■

The following two lemmas show that blocks of type $\mathcal{B}_1$ and of type $\mathcal{B}_2$ can be recognized in linear time:

(4.3.2) *It can be decided in* $O(|V(B)|)$ *time whether a given graph* B *is of the* $\mathcal{B}_1$ *type.*

Proof. We may assume that $|E(B)| \leq 2|V(B)|$, because if not, then it follows from (4.3.1) that B is not of the $\mathcal{B}_1$ type. Bodlaender **[8]** proved that, for any fixed k, finding a cycle of length at least k in a given graph H, if it exists, can be done in $O(k!2^k|V(H)|)$ time. The following algorithm uses Bodlaender's algorithm multiple times to recognize graphs of the $\mathcal{B}_1$ type.

(1) For $p = 8, 7, 6, 5$, do:

   Check if B contains a cycle of length p or more. If so, let F be the cycle and go to step (3).

(2) B does not contain a cycle of length 5 or larger, and hence B is not of the $\mathcal{B}_1$ type and we return NO.

(3) Let $k = |V(F)|$. If $k \in \{6, 8\}$, then B is not of the $\mathcal{B}_1$ type and we return NO. Let $f_1, f_2, \dots, f_k$

be the vertices of F in order. If $k = 7$, check that the 'inner edges' of F are as in the definition of $\mathcal{B}_1$. If not, B is not of the $\mathcal{B}_1$ type and we return NO.
For $i \in [k]$, do:

> Let A_i be the vertices in $V(B) \setminus V(F)$ that are adjacent to exactly f_{i-1} and f_{i+1}. If $|A_i| \geq 1$ and $\deg(f_i) \neq 2$, then B is not of the $\mathcal{B}_1$ type and we return NO.

If $\sum_{i=1}^{k} |A_i| + |V(F)| < |V(B)|$, then B is not of the $\mathcal{B}_1$ type and return NO.

(4) B is of the $\mathcal{B}_1$ type and we return YES.

It is not hard to verify that this algorithm takes $O(|V(B)|)$ time. This proves (4.3.2). ■

(4.3.3) *It can be decided in $O(|V(B)|)$ time whether a given graph B is of the $\mathcal{B}_2$ type.*

Proof. We may assume that $|E(B)| \leq 2|V(B)|$, because if not, then it follows from (4.3.1) that B is not of the $\mathcal{B}_2$ type. Clearly, it can be checked in constant time whether B is isomorphic to K_2, K_3, K_4, $K_{2,2}$ or $K_{2,2}^+$. So we may assume that B is either isomorphic to $K_{2,t}$ or $K_{2,t}^+$ for some $t \geq 3$, or B is not of the $\mathcal{B}_2$ type. Let $X \subseteq V(B)$ be the set of vertices of degree 2. If $|X| \neq |V(B)| - 2$, then B is not of the $\mathcal{B}_2$ type and we may stop. Otherwise, let $\{a_1, a_2\} = V(B) \setminus X$. We need to check that X is an stable set and X is complete to $\{a_1, a_2\}$. If so, then B is of the $\mathcal{B}_1$ type and we may stop. If not, then B is not of the $\mathcal{B}_2$ type and we may stop. Notice that, since $|E(B)| \leq 2|V(B)|$, the check above can be done in $O(|E(B)|)$ time. This proves (4.3.3). ■

This puts us in a position to prove Theorem 4.0.4.

Theorem 4.0.4. *It can be decided in $O(|V(G)|)$ time whether a graph G satisfies the LoP conditions.*

Proof. We may assume that G is connected. By Theorem 4.0.3 and Theorem 4.2.1, it suffices to check whether G admits the structure described in Theorem 4.2.1. We propose the following algorithm. Let $n = |V(G)|$ and $m = |E(G)|$. First we check that $m \leq 2n$, because otherwise G is not $\mathcal{C}$-free by (4.3.1) and we stop immediately. Now, construct the block decomposition $\{B_1, B_2, ..., B_q\}$ of G. This can, in general, be done in $O(n + m)$ time (see *e.g.* **[31]**). However, since we know that $m \leq 2n$, this step actually takes $O(n)$ time. For each block B_i, we test whether B_i is of the $\mathcal{B}_2$ type. This can be done $O(|V(B_i)|)$ time by (4.3.3). If G has more than one block that is not of the $\mathcal{B}_2$ type, then G is not $\mathcal{C}$-free and we stop. If we encounter no such block, then G is $\mathcal{C}$-free and we stop. So let B^* be the unique block that is not of the $\mathcal{B}_2$ type. It follows from (4.3.2) that it can be decided in $O(|V(B^*)|)$ time whether B^* is of the $\mathcal{B}_1$ type or not. If it is, then G is $\mathcal{C}$-free and we stop. If not, then G is not $\mathcal{C}$-free and we stop. This proves Theorem Theorem 4.0.4. ■

5

Claw-free graphs with strongly perfect complements: fractional and integral version

A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. We say that a graph G is *strongly perfect* if every induced subgraph H of G contains a stable set that meets every (inclusion-wise) maximal clique of H. Strongly perfect graphs were first studied by Berge and Duchet **[5]** as a special class of perfect graphs. They form a natural special class of perfect graphs in the following sense: every perfect graph (and hence each of its induced subgraphs) contains a stable set that meets every maximum cardinality clique. Strongly perfect graphs satisfy the stronger property that they contain a stable set meeting every inclusion-wise maximal clique.

An equivalent definition of strong perfection is: a graph G is strongly perfect if and only if for every induced subgraph H of G, there exists a function $w : V(H) \to \{0, 1\}$ such that $\sum_{v \in K} w(v) = 1$ for every maximal clique K of H. This definition leads to the following natural relaxation. We say that a graph G is *fractionally strongly perfect* if, for every induced subgraph H of G, there exists a function $w : V(H) \to [0, 1]$ such that $\sum_{v \in K} w(v) = 1$ for every maximal clique K of H. We say that a graph G is *fractionally co-strongly perfect* if G^c is fractionally strongly perfect. As it is this concept that we are interested in, we give the following equivalent definition of fractional co-strong perfection:

Definition. *A graph G is* fractionally co-strongly perfect *if and only if, for every induced subgraph H of G, there exists a function $w : V(H) \to [0, 1]$ such that*

$$\sum_{v \in S} w(v) = 1, \text{ for every maximal stable set } S \text{ of } H. \tag{5.1}$$

We call a function w that satisfies (5.1) a saturating vertex weighting *for H.*

A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. Claw-free graphs are a generalization of line graphs. We investigate graphs that are claw-free and that are fractionally co-strongly perfect. In Chapter 4, we characterized all line graphs that are fractionally co-strongly perfect.

The current chapter presents a generalization of that result to the setting of claw-free graphs. We will give a characterization of such graphs in terms of forbidden induced subgraphs. Wang **[57]** gave a characterization of claw-free graphs that are strongly perfect. As a corollary of our main theorem, we obtain a characterization of claw-free graphs that are strongly perfect in the complement; see Subsection 5.1.

Main results

Before stating our main theorem, we define the following three classes of graphs:

- $\mathcal{F}_1 = \{C_k \mid k = 6 \text{ or } k \geq 8\}$, where C_k is a cycle of length k;
- $\mathcal{F}_2 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$, where the $\mathcal{G}_i$'s are the graphs drawn in Figure 5.1(a);
- Let $\mathcal{H} = \{\mathcal{H}_1(k), \mathcal{H}_2(k), \mathcal{H}_3(k) \bigm| k \geq 0\}$, where $\mathcal{H}_i(k)$ is the graph $\mathcal{H}_i$ drawn in Figure 5.1(b) but whose 'wiggly' edge joining z and x is replaced by an induced k-edge-path. For $i \in \{1, 2, 3\}$, we call $\mathcal{H}_i(k)$ a *heft of type i with a rope of length k*. We call x the *end of the heft $\mathcal{H}_i(k)$*.

  Now let $i_1, i_2 \in \{1, 2, 3\}$ and let $k_1, k_2 \geq 0$ be integers. Let $H_1 = \mathcal{H}_{i_1}(k_1)$ and $H_2 = \mathcal{H}_{i_2}(k_2)$, and let x_1, x_2 be the end of heft H_1, H_2, respectively. Construct H from the disjoint union of H_1 and H_2 by deleting x_1 and x_2, and making the neighbors of x_1 complete to the neighbors of x_2. Then H is called a *skipping rope of type (i_1, i_2) of length $k_1 + k_2$*. Let $\mathcal{F}_3$ be the collection of skipping ropes. Figure 5.2 shows two examples of skipping ropes.

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. A graph G is *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to a graph in $\mathcal{F}$. We say that a graph G is *resolved* if at least one of the following is true:

(a) there exists $x \in V(G)$ that is complete to $V(G) \setminus \{x\}$; or

(b) G has a dominant clique; or

(c) G is not perfect and there exists $k \in \{2, 3\}$ such that every maximal stable set in G has size k.

We say that a graph G is *perfectly resolved* if every *connected* induced subgraph of G is resolved. In this chapter, we will prove the following theorem:

Theorem 5.0.4. *Let G be a claw-free graph. Then the following statements are equivalent:*

(i) *G is fractionally co-strongly perfect;*
(ii) *G is $\mathcal{F}$-free;*
(iii) *G is perfectly resolved.*

Wang **[57]** gave a characterization of claw-free graphs that are strongly perfect. Theorem 5.0.4 allows us to give a characterization of claw-free graphs that are strongly perfect *in the complement*. Specifically, we obtain the following induced subgraph characterization of claw-free graphs that are strongly perfect in the complement:

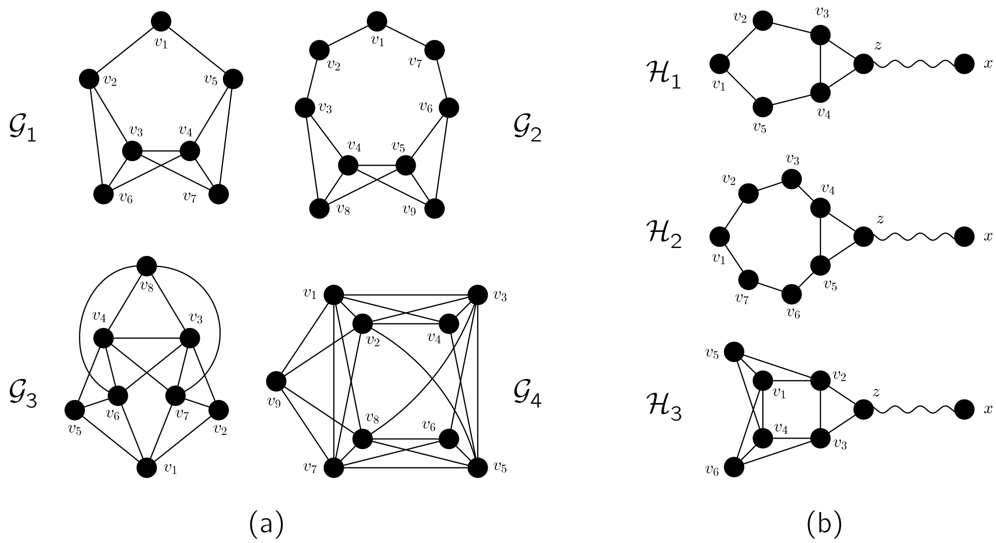


Figure 5.1: *Forbidden induced subgraphs for fractionally co-strongly perfect graphs. (a) The graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$. (b) Hefts $\mathcal{H}$ that are combined to construct skipping ropes.*

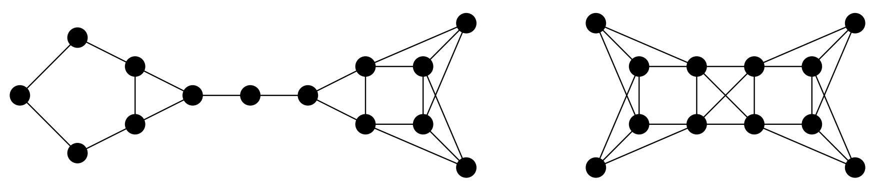


Figure 5.2: *Two examples of skipping ropes. Left: the skipping rope of type* $(1, 3)$ *of length* 3. *Right: the skipping rope of type* $(3, 3)$ *of length* 0.

Theorem 5.0.5. *Let G be a claw-free graph. G^c is strongly perfect if and only if G is perfect and no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even hole of length at least six, or a skipping rope of type* $(3, 3)$ *of length $k \geq 0$.*

Chudnovsky and Seymour **[14]** proved a structure theorem for claw-free graphs. The theorem roughly states that every claw-free graph is either of a certain 'basic' type or admits a so-called 'strip-structure'. In fact, **[14]** deals with slightly more general objects called 'claw-free trigraphs'. What is actually meant by 'basic trigraph' and 'a trigraph that admits strip-structure' will be explained in Section 5.2.

Organization of this chapter

This chapter is organized as follows. We will start in Section 5.1 by assuming the validity of Theorem 5.0.4 and proving Theorem 5.0.5. In Section 5.2, we introduce tools that we need throughout the chapter. We introduce the notion of a 'trigraph', which is a generalization of graph, and we will define what is meant by a basic trigraph and a trigraph that admits a strip-structure. In Section 5.3, we will prove the easy directions of Theorem 5.0.4. Specifically, we will prove that (i) implies (ii) in Theorem 5.0.4:

Theorem 5.0.6. *If G is fractionally co-strongly perfect, then G is $\mathcal{F}$-free.*

And we will prove that (iii) implies (i) in Theorem 5.0.4:

Theorem 5.0.7. *If G is perfectly resolved, then G is fractionally co-strongly perfect.*

The bulk of this chapter is devote to proving the hard direction of Theorem 5.0.4, *i.e.*, that (ii) implies (iii) in Theorem 5.0.4:

Theorem 5.0.8. *Every $\mathcal{F}$-free basic claw-free graph G is resolved.*

We will start in Section 5.4 by considering all basic claw-free graphs, dealing successively with 'antiprismatic graphs', 'long circular interval graphs', and 'three-cliqued claw-free graphs'. Sections 5.5, 5.6, 5.7 will deal with strip-structures.

5.1 Claw-free graphs with strongly perfect complements

Wang **[57]** gave a characterization of claw-free graphs that are strongly perfect. Theorem 5.0.4 allows us to give a characterization of claw-free graphs whose complement is strongly perfect:

Theorem 5.0.5. *Let G be a claw-free graph. G^c is strongly perfect if and only if G is perfect and no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even hole of length at least six, or a skipping rope of type $(3,3)$ of length $k \geq 0$.*

Proof of Theorem 5.0.5. For the 'only-if' direction, let G be a claw-free graph such that G^c is strongly perfect. Since G^c is strongly perfect, G is fractionally co-strongly perfect. Therefore, it follows from Theorem 5.0.4 that G is $\mathcal{F}$-free and, in particular, no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even cycle of length at least six, or a skipping rope of type $(3,3)$ of length $k \geq 0$. Moreover, it follows from Theorem 5 of **[5]** applied to G^c that G is perfect. This proves the 'only-if' direction.

For the 'if' direction, let G be a perfect claw-free graph such that no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even cycle of length at least six, or a skipping rope of type $(3,3)$ of length $k \geq 0$. Since G is perfect, by the strong perfect graph theorem **[17]**, G has no odd hole or odd antihole of length at least five as induced subgraph. Because all graphs in $\mathcal{F}$ other than $\mathcal{G}_4$, the even holes and the skipping ropes of type $(3,3)$ contain an induced cycle of length five or length seven, it follows that G is $\mathcal{F}$-free and hence, by Theorem 5.0.4, G is perfectly resolved. Now recall that a graph G^c is strongly perfect if and only if every induced subgraph of G has a dominant clique. We note that every disconnected induced subgraph of G has a dominant clique if and only if one of its components has a dominant clique. Therefore, it suffices to show that every connected induced subgraph of G

has a dominant clique. So suppose to the contrary that G has a minimal connected induced subgraph H such that H has no dominant clique. Since G is perfectly resolved, it follows that H is resolved. Since H is perfect and H has no dominant clique, it follows that H has a vertex x that is complete to $V(H) \setminus \{x\}$. Since H is minimal, $H - x$ has a dominant clique K. But this implies that $K \cup \{x\}$ is a dominant clique in H, a contradiction. This proves Theorem 5.0.5. ■

Theorem 5.0.5 states that if a claw-free graph is perfect and it is fractionally co-strongly perfect, then it is integrally co-strongly perfect. We conjecture that this is true in general:

Conjecture 5.1.1. *If G is perfect and fractionally strongly perfect, then G is strongly perfect.*

5.2 Tools

In this section, we introduce definitions, notation and important lemmas that we use throughout the chapter. As in **[14]**, it will be helpful to work with "trigraphs" rather than with graphs. We use the terminology defined in this section for graphs as well. The definitions should be applied to graphs by regarding graphs as trigraphs. For any integer $n \geq 1$, we denote by $[n]$ the set $\{1, 2, \dots, n\}$.

5.2.1 Claw-free graphs and trigraphs

A *trigraph* T consists of a finite set $V(T)$ of vertices, and a map $\theta_T : V(T) \to \{1, 0, -1\}$, satisfying:

- $\theta_T(v, v) = 0$, for all $v \in V(T)$;
- $\theta_T(u, v) = \theta_T(v, u)$, for all distinct $u, v \in V(T)$;
- for all distinct $u, v, w \in V(T)$, at most one of $\theta_T(u, v)$, $\theta_T(u, w) = 0$.

We call θ_T the *adjacency function* of T. For distinct $u, v \in V(T)$, we say that u and v are *strongly adjacent* if $\theta_T(u, v) = 1$, *strongly antiadjacent* if $\theta_T(u, v) = -1$, and *semiadjacent* if $\theta_T(u, v) = 0$. We say that u and v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. We denote by $F(T)$ the set of all pairs $\{u, v\}$ such that $u, v \in V(T)$ are distinct and semiadjacent. Thus a trigraph T is a graph if $F(T) = \emptyset$.

We say that u is a *(strong) neighbor* of v if u and v are (strongly) adjacent; u is a *(strong) antineighbor* of v if u and v are (strongly) antiadjacent. For distinct $u, v \in V(T)$ we say that $uv = \{u, v\}$ is an *edge*, a *strong edge*, an *antiedge*, a *strong antiedge*, or a *semiedge* if u and v are adjacent, strongly adjacent, antiadjacent, strongly antiadjacent, or semiadjacent, respectively. For disjoint sets $A, B \subseteq V(T)$, we say that A is *(strongly) complete to B* if every vertex in A is (strongly) adjacent to every vertex in B, and that A is *(strongly) anticomplete to B* if every vertex in A is (strongly) antiadjacent to every vertex in B. We say that A and B are *linked* if every vertex in A has a neighbor in B and every vertex in B has a neighbor in A. For $v \in V(T)$, let $N_T(v)$ denote the set of vertices adjacent to v, and let $N_T[v] = N_T(v) \cup \{v\}$. Whenever it is clear from the context what T is, we drop the subscript and write $N(v) = N_T(v)$ and $N[v] = N_T[v]$. For $X \subseteq V(T)$, we write $N(X) = (\cup_{x \in X} N(x)) \setminus X$ and $N[X] = N(X) \cup X$.

We say that a trigraph T' is a *thickening of* T if for every $v \in V(T)$ there is a nonempty subset $X_v \subseteq V(T')$, all pairwise disjoint and with union $V(T')$, satisfying the following:

(i) for each $v \in V(T)$, X_v is a strong clique of T';

(ii) if $u, v \in V(T)$ are strongly adjacent in T, then X_u is strongly complete to X_v in T';

(iii) if $u, v \in V(T)$ are strongly antiadjacent in T, then X_u is strongly anticomplete to X_v in T';

(iv) if $u, v \in V(T)$ are semiadjacent in T, then X_u is neither strongly complete nor strongly anticomplete to X_v in T'.

When $F(T') = \emptyset$ then we call T' regarded as a graph a *graphic thickening of* T.

For $X \subseteq V(T)$, we define the trigraph $T|X$ *induced on* X as follows. The vertex set of $T|X$ is X, and the adjacency function of $T|X$ is the restriction of θ_T to X^2. We call $T|X$ an *induced subtrigraph* of T. We define $T \setminus X = T|(V(T) \setminus X)$. We say that a graph G is a *realization of* T if $V(G) = V(T)$ and for distinct $u, v \in V(T)$, u and v are adjacent in G if u and v are strongly adjacent in T, u and v are nonadjacent in G if u and v are strongly antiadjacent in T, and u and v are either adjacent or nonadjacent in G if u and v are semiadjacent in T. We say that T contains a graph H as a *weakly induced subgraph* if there exists a realization of T that contains H as an induced subgraph. We mention the following easy lemma:

(5.2.1) *Let T be a trigraph and let H be a graph. If T contains H as a weakly induced subgraph, then every graphic thickening of T contains H as an induced subgraph.*

Proof. Let G be a graphic thickening of T. Since T contains H as a weakly induced subgraph, there exists a realization G' of T that contains H as an induced subgraph. Because every graphic thickening of T contains every realization of T as an induced subgraph, it follows that G contains H as an induced subgraph. This proves (5.2.1). ■

We say that a set $K \subseteq V(T)$ is a *(strong) clique* if the vertices in K are pairwise (strongly) adjacent. We say that a set $S \subseteq V(T)$ is a *(strong) stable set* if the vertices in S are pairwise (strongly) antiadjacent. A stable set S is called a *triad* if $|S| = 3$. T is said to be *claw-free* if T does not contain the claw as a weakly induced subgraph. A trigraph T is said to be $\mathcal{F}$-free if it does not contain any graph in $\mathcal{F}$ as a weakly induced subgraph. We state the following trivial result without proof:

(5.2.2) *Let T be a claw-free trigraph. Then no $v \in V(T)$ is complete to a triad in T.*

Let $p_1, p_2, \dots, p_k \in V(T)$ be distinct vertices. We say that $T|\{p_1, p_2, \dots, p_k\}$ of T is a *weakly induced path (from p_1 to p_k)* in T if, for $i, j \in [k], i < j$, p_i and p_j are adjacent if $j = i + 1$ and antiadjacent otherwise. Let $\{c_1, c_2, \dots, c_k\} \subseteq V(T)$. We say that $T|\{c_1, c_2, \dots, c_k\}$ is a *weakly induced cycle (of length k)* in T if for all distinct $i, j \in [k]$, c_i is adjacent to c_j if $|i-j| = 1 \pmod k$, and antiadjacent otherwise. We say that $T|\{c_1, c_2, \dots, c_k\}$ is a *semihole (of length k)* in T if for all distinct $i, j \in [k]$, c_i is adjacent to c_j if $|i-j| = 1 \pmod k$, and strongly antiadjacent otherwise. A vertex v in a trigraph T is *simplicial* if $N(v)$ is a strong clique. Notice that our definition of a simplicial vertex differs slightly from the definition used in **[14]**, because we allow v to be incident with a semiedge.

Finally, we say that a set $X \subseteq V(T)$ is a *homogeneous set in T* if $|X| \geq 2$ and $\theta_T(x, v) = \theta_T(x', v)$ for all $x, x' \in X$ and all $v \in V(T) \setminus X$. For two vertices $x, y \in V(T)$, we say that x is a *clone of y* if $\{x, y\}$ is a homogeneous set in T. In that case we say that x and y are *clones*.

5.2.2 Classes of trigraphs

Let us define some classes of trigraphs:

- **Line trigraphs**. Let H be a graph, and let T be a trigraph with $V(T) = E(H)$. We say that T is a *line trigraph* of H if for all distinct $e, f \in E(H)$:
  - if e, f have a common end in H then they are adjacent in T, and if they have a common end of degree at least three in H, then they are strongly adjacent in T;
  - if e, f have no common end in H then they are strongly antiadjacent in T.
- **Trigraphs from the icosahedron**. The icosahedron is the unique planar graph with twelve vertices all of degree five. Let it have vertices $v_0, v_1, \dots, v_{11}$ where for $1 \leq i \leq 10$, v_i is adjacent to v_{i+1}, v_{i+2} (reading subscripts modulo 10), and v_0 is adjacent to v_1, v_3, v_5, v_7, v_9, and v_{11} is adjacent to $v_2, v_4, v_6, v_8, v_{10}$. Let this graph be T_0, regarded as a trigraph. Let T_1 be obtained from T_0 by deleting v_{11}. Let T_2 be obtained from T_1 by deleting v_{10}, and possibly by making v_1 semiadjacent to v_4, or making v_6 semiadjacent to v_9, or both. Then each of T_0, T_1, and the several possibilities for T_2 is a *trigraph from the icosahedron*.
- **Long circular interval trigraphs**. Let Σ be a circle, and let $F_1, \dots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \dots, F_k$ share an endpoint, and no three of them have union Σ. Now let $V \subseteq \Sigma$ be finite, and let T be a trigraph with vertex set V in which, for distinct $u, v \in V$,
  - if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent;
  - if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

  Such a trigraph T is called a *long circular interval trigraph*.
- **Antiprismatic trigraphs**. Let T be a trigraph such that for every $X \subseteq V(T)$ with $|X| = 4$, $T|X$ is not a claw and there are at least two pairs of vertices in X that are strongly adjacent in T. Moreover, if $u, v \in V(T)$ are semiadjacent, then either

– neither of u, v is in a triad; or

– there exists $w \in V(T)$ such that $\{u, v, w\}$ is a triad, but there is no other triad that contains u or v.

Then, T is called an *antiprismatic trigraph*.

We will use the following structural result from **[14]**:

Theorem 5.2.3. (**7.2** in **[14]**) *Let G be a connected claw-free graph. Then, either G admits a nontrivial strip-structure, or G is the graphic thickening of one of the following trigraphs:*

(a) *a trigraph of the icosahedron, or*

(b) *an antiprismatic trigraph, or*

(c) *a long circular interval trigraph, or*

(d) *a trigraph that is the union of three strong cliques.*

We say that a claw-free trigraph T is *basic* if T satisfies one of the outcomes (a)-(d) of Theorem 5.2.3. Analogously, a claw-free graph G is said to be *basic* if G is a graphic thickening of a basic claw-free trigraph.

5.2.3 Three-cliqued claw-free trigraphs

Let T be a trigraph such that $V(T) = A \cup B \cup C$ and A, B, C are strong cliques. Then (T, A, B, C) is called a *three-cliqued trigraph*. We define the following types of three-cliqued claw-free trigraphs:

$\mathcal{TC}_1$: **A type of line trigraph**. Let v_1, v_2, v_3 be distinct nonadjacent vertices of a graph H, such that every edge of H is incident with one of v_1, v_2, v_3. Let v_1, v_2, v_3 all have degree at least three, and let all other vertices of H have degree at least one. Moreover, for all distinct $i, j \in [3]$, let there be at most one vertex different from v_1, v_2, v_3 that is adjacent to v_i and not to v_j in H. Let A, B, C be the sets of edges of H incident with v_1, v_2, v_3 respectively, and let T be a line trigraph of H. Then (G, A, B, C) is a three-cliqued claw-free trigraph. Let $\mathcal{TC}_1$ be the class of all such three-cliqued trigraphs such that every vertex is in a triad.

$\mathcal{TC}_2$: **Long circular interval trigraphs**. Let T be a long circular interval trigraph, and let Σ be a circle with $V(T) \subseteq \Sigma$, and $F_1, \dots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(T)$ with $|X| \leq 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(T)$. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(T) \subseteq L_1 \cup L_2 \cup L_3$. Then $(T, V(T) \cap L_1, V(T) \cap L_2, V(T) \cap L_3)$ is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_2$ the class of such three-cliqued trigraphs with the additional property that every vertex is in a triad.

$\mathcal{TC}_3$: **Near-antiprismatic trigraphs** . Let $n \geq 2$. Construct a trigraph T as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n + 1$ and $|C| = n$, say

$A = \{a_0, a_1, \dots, a_n\}$, $B = \{b_0, b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$. Adjacency is as follows. A, B, C are strong cliques. For $0 \le i, j \le n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \le i \le n$ and $0 \le j \le n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not specified so far are strongly antiadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \ge 2$. Let all adjacent pairs be strongly adjacent except:

- a_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $b_i \in X$;
- b_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $a_i \in X$;
- a_i is semiadjacent to b_i for at most one value of $i \in [n]$, and if so then $c_i \in X$.

Let the trigraph just constructed be T. Then $T' = T \setminus X$ is a *near-antiprismatic trigraph*. Let $A' = A \setminus X$ and define B', C' similarly; then (T', A', B', C') is a three-cliqued trigraph. We denote by $\mathcal{TC}_3$ the class of all such three-cliqued trigraphs with the additional property that every vertex is in a triad.

$\mathcal{TC}_4$: **Antiprismatic trigraphs**. Let T be an antiprismatic trigraph and let A, B, C be a partition of $V(T)$ into three strong cliques; then (T, A, B, C) is a three-cliqued claw-free trigraph. We denote the class of all such three-cliqued trigraphs by $\mathcal{TC}_4$. Note that in this case there may be vertices that are in no triads.

$\mathcal{TC}_5$: **Sporadic exceptions**. There are two types of sporadic exceptions:

(1) Let T be the trigraph with vertex set $\{v_1, \dots, v_8\}$ and adjacency as follows: v_i, v_j are strongly adjacent for $1 \le i < j \le 6$ with $j - i \le 2$; the pairs v_1v_5 and v_2v_6 are strongly antiadjacent; v_1, v_6, v_7 are pairwise strongly adjacent, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \dots, v_6$; the pairs v_1v_4 and v_3v_6 are semiadjacent, and v_2 is antiadjacent to v_5. Let $A = \{v_1, v_2, v_3\}$, $B = \{v_4, v_5, v_6\}$ and $C = \{v_7, v_8\}$. Let $X \subseteq \{v_3, v_4\}$; then $(T \setminus X, A \setminus X, B \setminus X, C)$ is a three-cliqued trigraph, and all its vertices are in triads.

(2) Let T be the trigraph with vertex set $\{v_1, \dots, v_9\}$, and adjacency as follows: the sets $A = \{v_1, v_2\}$, $B = \{v_3, v_4, v_5, v_6, v_9\}$ and $C = \{v_7, v_8\}$ are strong cliques; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and the adjacency between the pairs v_2v_4 and v_5v_7 is arbitrary. Let $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that

- v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$;
- v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$;
- if $v_4, v_5 \in X$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

Then $(T \setminus X, A, B \setminus X, C)$ is a three-cliqued trigraph.

We denote by $\mathcal{TC}_5$ the class of such three-cliqued trigraphs (given by one of these two constructions) with the additional property that every vertex is in a triad.

We say that a three-cliqued trigraph (T, A, B, C) is *basic* if $(T, A, B, C) \in \bigcup_{i=1}^{5} \mathcal{TC}_i$. If (T, A, B, C) is a three-cliqued trigraph, and $\{A', B', C'\} = \{A, B, C\}$, then (T, A', B', C') is also a three-cliqued trigraph, that we say is a *permutation* of (T, A, B, C). Let $n \geq 0$, and for $1 \leq i \leq n$, let (T_i, A_i, B_i, C_i) be a three-cliqued trigraph, where $V(T_1), \dots, V(T_n)$ are all nonempty and pairwise vertex-disjoint. Let $A = A_1 \cup \cdots \cup A_n$, $B = B_1 \cup \cdots \cup B_n$, and $C = C_1 \cup \cdots \cup C_n$, and let T be the trigraph with vertex set $V(T_1) \cup \cdots \cup V(T_n)$ and with adjacency as follows:

- for $1 \leq i \leq n$, $T|V(T_i) = T_i$;
- for $1 \leq i < j \leq n$, A_i is strongly complete to $V(T_j) \setminus B_j$; B_i is strongly complete to $V(T_j) \setminus C_j$; and C_i is strongly complete to $V(T_j) \setminus A_j$; and
- for $1 \leq i < j \leq n$, if $u \in A_i$ and $v \in B_j$ are adjacent then u, v are both in no triads; and the same applies if $u \in B_i$ and $v \in C_j$, and if $u \in C_i$ and $v \in A_j$.

In particular, A, B, C are strong cliques, and so (T, A, B, C) is a three-cliqued trigraph; we call the sequence (T_i, A_i, B_i, C_i), $i \in [n]$, a *worn hex-chain* for (T, A, B, C). When $n = 2$, we say that (T, A, B, C) is a *worn hex-join* of (T_1, A_1, B_1, C_1) and (T_2, A_2, B_2, C_2). Note also that every triad of T is a triad of one of $T_1, \dots, T_n$, and if each T_i is claw-free then so is T. If we replace the third condition above by the strengthening

- for $1 \leq i < j \leq n$, the pairs (A_i, B_j), (B_i, C_j) and (C_i, A_j) are strongly anticomplete,

then we call the sequence a *hex-chain* for (T, A, B, C). When $n = 2$, (T, A, B, C) is a *hex-join* of (T_1, A_1, B_1, C_1) and (T_2, A_2, B_2, C_2). We will use the following theorem, which is a corollary of **4.1** in **[14]**.

Theorem 5.2.4. *Every claw-free graph that is a graphic thickening of a three-cliqued trigraph is a graphic thickening of a trigraph that admits a worn hex-chain into terms, each of which is a permutation of a basic three-cliqued trigraph.*

5.2.4 Properties of long circular interval trigraphs

A graph G is said to be a *long circular interval graph* if G, regarded as a trigraph, is a long circular interval trigraph. We use a characterization of long circular interval graphs that was given by **1.1** in **[13]**. We need some more definitions. A *net* is a graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$, such that $\{a_1, a_2, a_3\}$ is a clique and a_i, b_i are adjacent for $i = 1, 2, 3$, and all other pairs are nonadjacent. An *antinet* is the complement graph of a net. A $(1,1,1)$-prism is a graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$, such that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are cliques, and a_i, b_i are adjacent for $i = 1, 2, 3$, and all other pairs are nonadjacent. Let T be a trigraph. A *center* for a weakly induced cycle C is a vertex in $V(T) \setminus V(C)$ that is complete to $V(C)$ and a weakly induced cycle C is *dominating* in T if every vertex in $V(T) \setminus V(C)$ has a neighbor in $V(C)$. Since every realization of a long circular interval trigraph is a long circular interval graph, the following lemma is a straightforward corollary of **1.1** in **[13]**.

(5.2.5) *Let T be a long circular interval graph. Then, T does not contain a claw, net, antinet or $(1,1,1)$-prism as a weakly induced subgraph, and every weakly induced cycle of length at least four is dominating and has no center.*

Notice that, although **1.1** in **[13]** gives necessary and sufficient conditions, the reverse implication of (5.2.5) is not true.

5.2.5 A structure theorem for nonbasic claw-free trigraphs

Let T be a trigraph such that $V(T) = A \cup B \cup C$ and A, B, C are strong cliques. Then (T, A, B, C) is called a *three-cliqued trigraph*. Let (T, A, B, C) be a three-cliqued claw-free trigraph, and let $z \in A$ be such that z is strongly anticomplete to $B \cup C$. Let V_1, V_2, V_3 be three disjoint sets of new vertices, and let T' be the trigraph obtained by adding V_1, V_2, V_3 to T with the following adjacencies:

(i) V_1 and $V_2 \cup V_3$ are strong cliques;

(ii) V_1 is strongly complete to $B \cup C$ and strongly anticomplete to A;

(iii) V_2 is strongly complete to $A \cup C$ and strongly anticomplete to B;

(iv) V_3 is strongly complete to $A \cup B$ and strongly anticomplete to C.

The adjacency between V_1 and $V_2 \cup V_3$ is arbitrary. It follows that T' is claw-free, and z is a simplicial vertex of it. In this case we say that $(T', \{z\})$ is a *hex-expansion* of (T, A, B, C).

A *multigraph* H consists of a finite set $V(H)$, a finite set $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (*i.e.*, a subset of $V(H) \times E(H)$) such that every $F \in E(H)$ is incident with two members of $V(H)$ which are called the *endpoints* of F. For $F \in E(H)$, $\bar{F} = \{u, v\}$ where u, v are the two endpoints of F.

Let T be a trigraph. A *strip-structure* (H, η) of T consists of a multigraph H with $E(H) \neq \emptyset$, and a function η mapping each $F \in E(H)$ to a subset $\eta(F)$ of $V(T)$, and mapping each pair (F, h) with $F \in E(H)$ and $h \in \bar{F}$ to a subset $\eta(F, h)$ of $\eta(F)$, satisfying the following conditions.

(a) The sets $\eta(F)$ $(F \in E(H))$ are nonempty and pairwise disjoint and have union $V(T)$.

(b) For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \bar{F}$ is a strong clique of T.

(c) For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in T, then there exists $h \in \bar{F}_1 \cap \bar{F}_2$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.

(There is a fourth condition, but we do not need it here.) Let (H, η) be a strip-structure of a trigraph T, and let $F \in E(H)$, where $\bar{F} = \{h_1, h_2\}$. Let v_1, v_2 be two new vertices. Let $Z = \{v_i \mid i \in [2], \eta(F, h_i) \neq \emptyset\}$ and let J be the trigraph obtained from $T|\eta(F)$ by adding the vertices in Z, where $v_i \in Z$ is strongly complete to $\eta(F, h_i)$ and strongly anticomplete to all other vertices of J. Then (J, Z) is called the *strip of* (H, η) *at* F. (In the strip-structures that we are interested in, for every $F \in E(H)$ with $\bar{F} = \{h_1, h_2\}$, at least one of $\eta(F, h_1)$, $\eta(F, h_2)$ will be nonempty and therefore $1 \leq |Z| \leq 2$.)

Next, we list the classes of strips (T, Z) that we need for the structure theorem. We call the corresponding sets of pairs (T, Z) $\mathcal{Z}_1$-$\mathcal{Z}_{15}$.

$\mathcal{Z}_1$: Let T be a trigraph with vertex set $\{v_1, \dots, v_n\}$, such that for $1 \leq i < j < k \leq n$, if v_i, v_k are adjacent then v_j is strongly adjacent to both v_i, v_k. We call T a linear interval trigraph. (Every linear interval trigraph is also a long circular interval trigraph.) Also, let $n \geq 2$ and let v_1, v_n be strongly antiadjacent, and let there be no vertex adjacent to both v_1, v_n, and no vertex semiadjacent to either v_1 or v_n. Let $Z = \{v_1, v_n\}$.

$\mathcal{Z}_2$: Let $n \geq 2$. Construct a trigraph T' as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \dots, a_n\}$, $B = \{b_0, b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$. Adjacency is as follows. A, B, C are strong cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not specified so far are strongly antiadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let all adjacent pairs be strongly adjacent except:

- a_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $b_i \in X$
- b_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $a_i \in X$
- a_i is semiadjacent to b_i for at most one value of $i \in [n]$, and if so then $c_i \in X$

Let the trigraph just constructed be T' and let $T = T' \setminus X$. Let a_0 be strongly antiadjacent to b_0, and let $Z = \{a_0, b_0\}$.

$\mathcal{Z}_3$: Let H be a graph, and let h_1-h_2-h_3-h_4-h_5 be the vertices of a path of H in order, such that h_1, h_5 both have degree one in H, and every edge of H is incident with one of h_2, h_3, h_4. Let T be obtained from a line trigraph of H by making the edges h_2h_3 and h_3h_4 of H (vertices of T) either semiadjacent or strongly antiadjacent to each other in T. Let $Z = \{h_1h_2, h_4h_5\}$.

$\mathcal{Z}_4$: Let T be the trigraph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ and adjacency as follows: $\{a_0, a_1, a_2\}$, $\{b_0, b_1, b_2, b_3\}$, $\{a_2, c_1, c_2\}$ and $\{a_1, b_1, c_2\}$ are strong cliques; b_2, c_1 are strongly adjacent; b_2, c_2 are semiadjacent; b_3, c_1 are semiadjacent; and all other pairs are strongly antiadjacent. Let $Z = \{a_0, b_0\}$.

$\mathcal{Z}_5$: Let T' be the trigraph with vertex set $\{v_1, \dots, v_{13}\}$, with adjacency as follows. v_1-$\dots$-v_6-v_1 is a hole in T' of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5 and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$; and v_{13} is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$. No other pairs are adjacent, and all adjacent pairs are strongly adjacent except possibly for v_7, v_8 and v_9, v_{10} . (Thus the pair $v_7 v_8$ may be strongly adjacent, semiadjacent or strongly antiadjacent; the pair $v_9 v_{10}$ is either strongly adjacent or semiadjacent.) Let $T = T' \setminus X$, where $X \subseteq \{v_7, v_{11}, v_{12}, v_{13}\}$. Let v_7, v_8 be strongly antiadjacent in H, and let $Z = \{v_7, v_8\} \setminus X$.

$\mathcal{Z}_6$: Let T be a long circular interval trigraph, and let $\Sigma, F_1, \dots, F_k$ be as in the corresponding definition. Let $z \in V(T)$ belong to at most one of $F_1, \dots, F_k$, and not be an endpoint of any of $F_1, \dots, F_k$. Then z is a simplicial vertex of T; let $Z = \{z\}$.

$\mathcal{Z}_7$: Let H be a graph with seven vertices $h_1, \dots, h_7$, in which h_7 is adjacent to h_6 and to no other vertex, h_6 is adjacent to at least three of $h_1, \dots, h_5$, and there is a cycle with vertices h_1-h_2-$\dots$-h_5-h_1 in order. Let $J(H)$ be the graph obtained from the line graph of H by adding one new vertex, adjacent precisely to those members of $E(H)$ that are not incident with h_6 in H. Then $J(H)$ is a claw-free graph. Let T be either $J(H)$ (regarded as a trigraph), or (in the case when h_4, h_5 both have degree two in H), the trigraph obtained from $J(H)$ by making the vertices $h_3h_4, h_1h_5 \in V(J(H))$ semiadjacent. Let e be the edge h_6h_7 of H, and let $Z = \{e\}$.

$\mathcal{Z}_8$: Let $n \geq 2$. Construct a trigraph T as follows. Its vertex set is the disjoint union of four sets A, B, C and $\{d_1, \dots, d_5\}$, where $|A| = |B| = |C| = n$, say $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$, and $C = \{c_1, \dots, c_n\}$. Let $X \subseteq A \cup B \cup C$ with $|X \cap A|, |X \cap B|, |X \cap C| \leq 1$. Adjacency is as follows: A, B, C are strong cliques; for $1 \leq i, j \leq n$, a_i, b_j are adjacent if and only if $i = j$, and c_i is strongly adjacent to a_j if and only if $i \neq j$, and c_i is strongly adjacent to b_j if and only if $i \neq j$. Moreover,

- a_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $b_i \in X$;
- b_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $a_i \in X$;
- a_i is semiadjacent to b_i for at most one value of $i \in [n]$, and if so then $c_i \in X$;
- no two of $A \setminus X$, $B \setminus X$, $C \setminus X$ are strongly complete to each other.

Also, d_1 is strongly complete to $A \cup B \cup C$; d_2 is strongly complete to $A \cup B$, and either semiadjacent or strongly adjacent to d_1; d_3 is strongly complete to $A \cup \{d_2\}$; d_4 is strongly complete to $B \cup \{d_2, d_3\}$; d_5 is strongly adjacent to d_3, d_4; and all other pairs are strongly antiadjacent. Let the trigraph just constructed be T'. Let $T = T \setminus X$ and $Z = \{d_5\}$.

$\mathcal{Z}_9$: Let T have vertex set partitioned into five sets $\{z\}, A, B, C, D$, with $|A| = |B| > 0$, say $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ where $n \geq 1$, such that

- $\{z\} \cup D$ is a strong clique and z is strongly antiadjacent to $A \cup B \cup C$,
- $A \cup C$ and $B \cup C$ are strong cliques,

- for $1 \le i \le n$, a_i, b_i are antiadjacent, and every vertex in D is strongly adjacent to exactly one of a_i, b_i and strongly antiadjacent to the other, and
- for $1 \le i < j \le n$, $\{a_i, b_i\}$ is strongly complete to $\{a_j, b_j\}$.

(The adjacency between C and D is arbitrary.) Let $Z = \{z\}$.

$\mathcal{Z}_{10}$: Let T' be the trigraph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2, d\}$ and adjacency as follows: $A = \{a_0, a_1, a_2, d\}$, $B = \{b_0, b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$ and $\{a_1, b_1, c_2\}$ are strong cliques; a_2 is strongly adjacent to b_0 and semiadjacent to b_1; b_2, c_2 are semiadjacent; b_2, c_1 are strongly adjacent; b_3, c_1 are either semiadjacent or strongly adjacent; b_0, d are either semiadjacent or strongly adjacent; and all other pairs are strongly antiadjacent. Then (G, A, B, C) is a three-cliqued trigraph (not claw-free) and a_0 is a simplicial vertex of T'. Let $X \subseteq \{a_2, b_2, b_3, d\}$ such that either $a_2 \in X$ or $\{b_2, b_3\} \subseteq X$, let $Z = \{a_0\}$, and let (T, Z) be a hex-expansion of $(T' \setminus X, A \setminus X, B \setminus X, C)$.

$\mathcal{Z}_{11}$: Let $n \ge 2$. Construct a trigraph T' as follows. Its vertex set is the disjoint union of four sets $\{z\}, A, B, C$, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \dots, a_n\}$, $B = \{b_0, b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$. Adjacency is as follows. A, B, C are strong cliques. z is strongly complete to A and strongly anticomplete to $B \cup C$. For $0 \le i, j \le n$ with $(i, j) \ne (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \le i \le n$ and $0 \le j \le n$ let c_i be adjacent to a_j, b_j if and only if $i \ne j \ne 0$. a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not specified so far are strongly antiadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{b_0\}$ with $|C \setminus X| \ge 2$. Let all adjacent pairs be strongly adjacent except:

- a_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $b_i \in X$
- b_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $a_i \in X$
- a_i is semiadjacent to b_i for at most one value of $i \in [n]$, and if so then $c_i \in X$

Let the trigraph just constructed be T'. Let $Z = \{z\}$ and let (T, Z) be a hex-expansion of $(T' \setminus X, (A \setminus X) \cup Z, B \setminus X, C \setminus X)$.

$\mathcal{Z}_{12}$: Let T' be the trigraph with vertex set $\{v_1, \dots, v_9\}$, and adjacency as follows: the sets $A = \{v_1, v_2\}$, $B = \{v_3, v_4, v_5, v_6, v_9\}$ and $C = \{v_7, v_8\}$ are strong cliques; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and the adjacency between the pairs v_2v_4 and v_5v_7 is arbitrary. Let $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that

- v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$;
- v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$;
- if $v_4, v_5 \in X$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

Let T'' be the trigraph obtained from T' be adding a new vertex z. that is strongly complete to B. Let $Z = \{z\}$. Then $(T'' \setminus X, (B \cup Z) \setminus X, C, A)$ is a three-cliqued trigraph. Let (T, Z) be a hex-expansion of $(T'' \setminus X, (B \cup Z) \setminus X, C, A)$.

$\mathcal{Z}_{13}$: Let T' be a long circular interval trigraph such that every vertex of $'T$ is in a triad, and let Σ be a circle with $V(T') \subseteq \Sigma$, and $F_1, \dots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(T)$ with $|X| \leq 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(T)$. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(T') \subseteq L_1 \cup L_2 \cup L_3$. Then $(T', V(T') \cap L_1, V(T') \cap L_2, V(T') \cap L_3)$ is a three-cliqued claw-free trigraph. Let $z \in L_1$ belong to the interior of F_1. Thus, z is a simplicial vertex of T'. Let $Z = \{z\}$ and let (T, Z) be a hex-expansion of $(T', V(T') \cap L_1, V(T') \cap L_2, V(T') \cap L_3)$.

$\mathcal{Z}_{14}$: Let v_0, v_1, v_2, v_3 be distinct vertices of a graph H', such that: v_1 is the only neighbor of v_0 in H'; every vertex of H' different from v_0, v_1, v_2, v_3 is adjacent to both v_2, v_3, and at most one of them is nonadjacent to v_1; v_1, v_2, v_3 are pairwise nonadjacent, and each has degree at least three. For $i = 1, 2, 3$, let A_i be the set of edges of H' incident with v_i, and let z be the edge $v_0 v_1$. Let T' be a line trigraph of H'; thus (T', A_1, A_2, A_3) is a three-cliqued claw-free trigraph, and z is a simplicial vertex of T' . Let $Z = \{z\}$, and let (T, Z) be a hex-expansion of (T', A_1, A_2, A_3).

$\mathcal{Z}_{15}$: Let T' be the trigraph with vertex set $\{v_1, \dots, v_8\}$ and adjacency as follows: v_i, v_j are strongly adjacent for $1 \leq i < j \leq 6$ with $j - i \leq 2$; the pairs $v_1 v_5$ and $v_2 v_6$ are strongly antiadjacent; $\{v_1, v_6, v_7\}$ is a strong clique, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \dots, v_6$; the pairs $v_1 v_4$ and $v_3 v_6$ are semiadjacent, and v_2 is antiadjacent to v_5. Let $A = \{v_1, v_2, v_3\}$, $B = \{v_4, v_5, v_6\}$ and $C = \{v_7, v_8\}$. Let $X \subseteq \{v_3, v_4\}$; then $(T' \setminus X, A \setminus X, B \setminus X, C)$ is a three-cliqued trigraph and all its vertices are in triads. Let $Z = \{v_8\}$ and let (T, Z) be a hex-expansion of $(T' \setminus X, A \setminus X, B \setminus X, C)$.

Let $\mathcal{Z}_0 = \mathcal{Z}_1 \cup \dots \cup \mathcal{Z}_{15}$. We say that a claw-free trigraph T is called *basic* if T is a trigraph from the icosahedron, an antiprismatic trigraph, a long circular interval trigraph, or a trigraph that is the union of three strong cliques (we refer to **[18]** for their definitions), and T is called *nonbasic* otherwise. Analogously, a claw-free graph G is *basic* if G is the graphic thickening of a basic claw-free trigraph T and G is called *nonbasic* otherwise.

Let $F \in E(H)$ and let (J, Z) be the strip of (H, η) at F. We say that (J, Z) is a *spot* if $\eta(F) = \eta(F, u) = \eta(F, v)$ and $|\eta(F)| = 1$. Let J' be a thickening of J and, for $v \in V(J)$, let X_v be the strong clique in J' that corresponds to v. Let $Z' = \bigcup_{z \in Z} X_z$. If $|X_z| = 1$ for each $z \in Z$, then we say that (J', Z') is a *thickening of* (J, Z).

A strip-structure (H, η) is *nontrivial* if $|E(H)| \geq 2$. We say that a strip-structure (H, η) is *proper* if all of the following hold:

(1) (H, η) is nontrivial;

(2) for each strip (J, Z), either

  (a) (J, Z) is a spot, or
  (b) (J, Z) is a thickening of a member of $\mathcal{Z}_0$;

(3) for every $F \in E(H)$, if the strip of (H, η) at F is a thickening of a member of $\mathcal{Z}_6 \cup \mathcal{Z}_7 \cup \ldots \cup \mathcal{Z}_{15}$, then, at least one of the vertices in $\bar{F}$ has degree 1.

We note that in the definition of a strip-structure (H, η) given in **[14]**, the multigraph H is actually a hypergraph. In this hypergraph, however, every hyperedge has cardinality either one or two. We may replace every hyperedge F of cardinality one by a new vertex, z say, and a new edge F' with $\{u, z\}$, where u is the unique vertex in $\bar{F}$, and setting $\eta(F') = \eta(F)$, $\eta(F', u) = \eta(F, u)$, and $\eta(F, z) = \emptyset$. Thus, we may regard this hypergraph as a multigraph. With this observation in mind, the following theorem is an easy corollary of the main result of **[14]**.

(5.2.6) (**[14]**) *Every connected nonbasic claw-free graph is a graphic thickening of a claw-free trigraph that admits a proper strip-structure.*

5.2.6 Resolved graphs and trigraphs; finding dominant cliques

We say that an $\mathcal{F}$-free claw-free trigraph T is *resolved* if every $\mathcal{F}$-free thickening of T is resolved. We state a number of useful lemmas for concluding that a trigraph is resolved. Let T be a trigraph. For a vertex $x \in V(T)$, we say that a stable set $S \subseteq V(T)$ *covers* x if x has a neighbor in S. For a strong clique $K \subseteq V(T)$, we say that a stable set $S \subseteq V(T)$ *covers* K if S covers every vertex in K. We say that a strong clique $K \subseteq V(T)$ is a *dominant clique* if T contains no stable set $S \subseteq V(T) \setminus K$ such that S covers K. It is easy to see that this definition of a dominant clique, when applied to a graph, coincides with our earlier definition of a dominant clique for a graph.

(5.2.7) *Let T be a trigraph and suppose that K is a dominant clique in T. Then, T is resolved.*

Proof. Let G be a graphic thickening of T. For $v \in V(T)$, let X_v denote the clique in G corresponding to v. We claim that $K' = \bigcup_{z \in K} X_z$ is a dominant clique in G. For suppose not. Then there exists a maximal stable set $S' \subseteq V(G)$ such that $S' \cap K' = \emptyset$. Write $S' = \{s'_1, \ldots, s'_p\}$, where $p = |S'|$, and let $s_i \in V(T)$ be such that $s'_i \in X_{s_i}$. Let $S = \{s_1, \ldots, s_p\}$. We claim that S covers K, contrary to the fact that K is a dominant clique in T. Since S' is a stable set in G, it follows that S is a stable set in T. Now let $w \in K$ and let $w' \in X_w$. Since S' is maximal and $S' \cap K' = \emptyset$, it follows that w' has a neighbor $s' \in S'$. Let $s \in V(T)$ be such that $s' \in X_s$. It follows that w is adjacent to s. This proves that every $w \in K$ has a neighbor in S and, hence, S covers K, which proves (5.2.7). ■

Notice that if G is a graphic thickening of some trigraph T and T has no dominant clique, then this does not necessarily imply that G has no dominant clique (consider, for example, a two-vertex trigraph where the two vertices are semiadjacent). The following lemma gives another way of finding a dominant clique:

(5.2.8) *Let T be a trigraph, let A and B be nonempty disjoint strong cliques in T and suppose that A is strongly anticomplete to $V(T) \setminus (A \cup B)$. Then, T is resolved.*

Proof. Let G be a graphic thickening of T. For $v \in V(T)$, let X_v be the corresponding clique in G. Let $Y = \bigcup_{a \in A} X_a$ and $Z = \bigcup_{b \in B} X_b$. Let $Z' \subseteq Z$ be the set of vertices in Z that are complete to Y. We claim that $K = Y \cup Z'$ is a dominant clique in G. For suppose that S is a maximal stable set in G such that $S \cap K = \emptyset$. First notice that every $y \in Y$ has a neighbor in $(Z \setminus Z') \cap S$, because, if not, then we may add y to S and obtain a larger stable set. In particular, $(Z \setminus Z') \cap S \neq \emptyset$ and, since Z is a clique, $|S \cap (Z \setminus Z')| = 1$. But now the unique vertex z in $(Z \setminus Z') \cap S$ is complete to Y, contrary to the fact that $z \notin Z'$. This proves (5.2.8). ■

By letting $|A| = 1$ in (5.2.8), we obtain the following immediate result that we will use often:

(5.2.9) *Let T be a trigraph and let $v \in V(T)$ be a simplicial vertex. Then, T is resolved.*

Next, we have a lemma that deals with trigraphs with no triads:

(5.2.10) *Let T be a trigraph with no triad. Then, T is resolved.*

Proof. Let G be a graphic thickening of T. Since T has no triad, it follows that $\alpha(G) \leq 2$. If some vertex $v \in V(G)$ is complete to $V(G) \setminus \{v\}$, then G is resolved. So we may assume that no such vertex exists. It follows that there is no maximal stable set of size one and, hence, every maximal stable set has size two. If G is imperfect, then G is resolved. So we may assume that G is perfect. From this, since G^c has no triangles, it follows that G^c is bipartite and thus G is the union of two cliques. But now, it follows from (5.2.8) that G has a dominant clique and, therefore, G is resolved. This proves (5.2.10). ■

Let T be a trigraph, and suppose that K_1 and K_2 are disjoint nonempty strong cliques. We say that (K_1, K_2) is a *homogeneous pair of cliques* in T if, for $i = 1, 2$, every vertex in $V(T) \setminus (K_1 \cup K_2)$ is either strongly complete or strongly anticomplete to K_i. For notational convenience, for a weakly induced path $P = p_1\text{-}p_2\text{-}\ldots\text{-}p_{k-1}\text{-}p_k$, we define the *interior P^* of P* by $P^* = p_2\text{-}p_3\text{-}\cdots\text{-}p_{k-2}\text{-}p_{k-1}$.

(5.2.11) *Let T be an $\mathcal{F}$-free claw-free trigraph. Let (K_1, K_2) be a homogeneous pair of cliques in T such that K_1 is not strongly complete and not strongly anticomplete to K_2. For $\{i, j\} = \{1, 2\}$, let $N_i = N(K_i) \setminus N[K_j]$ and $M = V(T) \setminus (N[K_1] \cup N[K_2])$. If there exists a weakly induced path P between antiadjacent $v_1 \in N_1$ and $v_2 \in N_2$ such that $V(P^*) \subseteq M$ and $|V(P)| \geq 3$, then T is resolved.*

Proof. Let G be an $\mathcal{F}$-free graphic thickening of T. For $v \in V(T)$, let X_v denote the corresponding clique in G. Let $K_1' = \bigcup_{v \in K_1} X_v$ and define K_2', N_1', N_2', M' analogously. Let $Z' = (N(K_1') \cap N(K_2')) \setminus (K_1' \cup K_2')$. Since (K_1, K_2) is a homogeneous pair of cliques, it follows that, for $\{i, j\} = \{1, 2\}$, N_i' is complete to K_i' and anticomplete to K_j', and Z' is complete to $K_1' \cup K_2'$. Hence, from the fact that K_1' is not anticomplete to K_2' and the fact that G is claw-free, it follows that N_1' and N_2' are cliques. Z' is anticomplete to M', because if $z \in Z'$ has a neighbor $u \in M'$, then let $a \in K_1'$, $b \in K_2'$ be nonadjacent and observe that z is complete to the triad $\{a, b, u\}$, contrary to (5.2.2). We start with the following claim.

(i) *Suppose that there exist $a_1, a_2 \in K_1'$, $b \in K_2'$ such that b is adjacent to a_1 and nonadjacent to a_2. Let $x_1 \in N_1', x_2 \in N_2'$ be nonadjacent such that there is an induced path Q between x_1 and x_2 that satisfies $V(Q^*) \subseteq M'$. Then $|V(Q)| \in \{3, 5\}$ and Z' is complete to N_1'.*

Since b-a_1-x_1-Q^*-x_2-b is an induced cycle of length $|V(Q)| + 2$ and G contains no induced cycle of length 6 or at least 8, it follows that $|V(Q)| \in \{3, 5\}$. We may assume that $Z' \neq \emptyset$, otherwise we are done. We first claim that Z' is complete to x_1. For suppose that $z \in Z'$ is nonadjacent to x_1. If z is nonadjacent to x_2, then z-a_2-x_1-Q^*-x_2-b-z is an induced cycle of length $|V(Q)| + 3 \in \{6, 8\}$, a contradiction. Therefore, z is adjacent to x_2. But now, $G|(V(Q) \cup \{a_1, a_2, b, z\})$ is isomorphic to $\mathcal{G}_1$ if $|V(Q)| = 3$ or $\mathcal{G}_2$ if $|V(Q)| = 5$, a contradiction. This proves that Z' is complete to x_1.

Now let $p \in N_1'$ and suppose that p is nonadjacent to some $z \in Z'$. Let $u \in V(Q)$ be the unique neighbor of x_1 in Q. Because x_1 is complete to $\{p, u, z\}$, it follows from (5.2.2) that $\{p, u, z\}$ is not a triad and hence p is adjacent to u. If p is nonadjacent to x_2, then possibly by shortcutting Q, there is a path between nonadjacent p and x_2, and it follows from the previous argument that Z' is complete to p, a contradiction. It follows that p is adjacent to x_2. If $|V(Q)| = 5$, then u is nonadjacent to x_2 and hence p is complete to the triad $\{a_1, x_2, u\}$, contrary to (5.2.2). It follows that $|V(Q)| = 3$. Now, if z is nonadjacent to x_2, then $G|\{z, x_1, p, x_2, b, a_2, u\}$ is isomorphic to $\mathcal{G}_1$. Thus, z is adjacent to x_2. But now, $G|\{a_1, b, x_2, u, x_1, p, z, a_2\}$ is isomorphic to $\mathcal{G}_3$. This proves **(i)**. □

Let $P = p_1$-p_2-…-p_{k-1}-p_k be a weakly induced path between antiadjacent $p_1 \in N_1$ and $p_2 \in N_2$ such that $V(P^*) \subseteq M$ and $|V(P)| \geq 3$. For $i \in [k]$, let $p_i' \in X_{p_i}$ such that p_1'-…-p_k' is an induced path in G. It follows that $p_1' \in N_1'$, $p_k' \in N_2'$, and $V((P')^*) \subseteq M'$. We claim the following:

(ii) *Z' is a clique.*

Because K_1' is not complete and not anticomplete to K_2', we may assume from the symmetry that there exist $a_1, a_2 \in K_1'$ and $b \in K_2'$ such that b is adjacent to a_1 and nonadjacent to a_2. It follows from **(i)** that Z' is complete to p_1'. Let $u \in V(P')$ be the unique neighbor of p_1' in P'. If $z_1, z_2 \in Z'$ are nonadjacent, then p_1' is complete to the triad $\{z_1, z_2, u\}$, contrary to (5.2.2). This proves **(ii)**. □

The last claim deals with an easy case:

(iii) *If some vertex in K_1' is complete to K_2', then the lemma holds.*

Suppose that $a_1 \in K_1'$ is complete to K_2'. First observe that no vertex in K_1' has both a neighbor and a nonneighbor in K_2', because if $a_2 \in K_1'$ has a neighbor $b_1 \in K_2'$ and a nonneighbor $b_2 \in K_2'$, then $G|(V(P') \cup \{a_1, a_2, b_1, b_2\})$ is isomorphic to $\mathcal{G}_1$ if $|V(P')| = 3$ and to $\mathcal{G}_2$ if $|V(P')| = 5$. It follows that every vertex in K_1' is either complete or anticomplete to K_2'. Since K_1' is not complete to K_2', it follows that there exists $a_2 \in K_1'$ that is is anticomplete to K_2'. Now it follows from **(i)** that Z' is complete to N_1'. Thus, a_2 is a simplicial vertex and the lemma holds by (5.2.9). This proves **(iii)**. □

It follows from **(iii)** and the symmetry that we may assume that, for $\{i, j\} = \{1, 2\}$, no vertex in K'_i is complete to K'_j. Thus, it follows from **(i)** and the fact that K'_1 is not complete and not anticomplete to K'_2 that Z' is complete to $N'_1 \cup N'_2$. We claim that $K = K'_1 \cup Z' \cup N'_1$ is a dominant clique. For suppose not. Then there exists a maximal stable set S in G such that $S \cap K = \emptyset$. Let $a \in K'_1$. Since $N(a) \subseteq K \cup K'_2$, it follows that a has a neighbor in $S \cap K'_2$, because otherwise we may add a to S and obtain a larger stable set. In particular, $S \cap K'_2 \neq \emptyset$ and, since K'_2 is a clique, $|S \cap K'_2| = 1$. But now, the unique vertex v in $S \cap K'_2$ is complete to K'_1, a contradiction. This proves that K is a dominant clique, thus proving (5.2.11). ■

We note the following special case of (5.2.11), in which the two strong cliques of the homogeneous pair of cliques have cardinality one:

(5.2.12) *Let T be an $\mathcal{F}$-free claw-free trigraph and suppose that T contains a weakly induced cycle c_1-c_2-…-c_k-c_1 with $k \geq 5$ and such that $c_1c_2 \in F(T)$. Then, T is resolved.*

Proof. Since $c_1c_2 \in F(T)$, it follows from the definition of a trigraph that, for $i \in [2]$, every vertex in $V(T) \setminus \{c_1, c_2\}$ is either strongly adjacent or strongly antiadjacent to c_i. Thus, $(\{c_1\}, \{c_2\})$ is a homogeneous pair of cliques in T. Moreover, c_3-…-c_k is a weakly induced path that meets the conditions of (5.2.11). Thus, T is resolved by (5.2.11). This proves (5.2.12). ■

The following lemma states that we may assume that trigraphs do not have clones.

(5.2.13) *Let T be a trigraph and suppose that $v, w \in V(T)$ are clones of each other. If $T \setminus v$ is resolved, then T is resolved.*

Proof. First notice that it follows from the definitions of trigraphs and clones that v and w only have strong neighbors and strong antineighbors. Let G be a graphic thickening of T, and for all $u \in V(T)$ let X_u be the clique in G corresponding to u. Since $T \setminus v$ is resolved, we have that $G \setminus X_v$ is resolved, and thus there are three possibilities. First, suppose that $G \setminus X_v$ contains a vertex z that is complete to $V(G \setminus X_v) \setminus \{z\}$. Since v and w are clones, it follows that z is complete to X_v, and hence z is complete to $V(G) \setminus \{z\}$. Therefore, G is resolved. Next, suppose that $G \setminus X_v$ has a dominant clique K. Notice that either $X_w \subseteq K$ or $X_w \cap K = \emptyset$, because otherwise K is not a dominant clique. Let $K' = K \cup \{X_v\}$ if $X_w \subseteq K$, and let $K' = K$ otherwise. We claim that K' is a dominant clique in G. For suppose there exists a maximal stable set S such that $S \cap K' = \emptyset$. If $X_v \cap S = \emptyset$, then clearly, S is a maximal stable set in $G \setminus X_v$ with $S \cap K = \emptyset$, contrary to the fact that K is a dominant clique in $G \setminus X_v$. Therefore, $X_v \cap S \neq \emptyset$ and hence $K' = K$. Since v and w are clones in T, the set $S' = (S \setminus \{X_v\}) \cup \{w'\}$, where $w' \in X_w$, is a stable set in $G \setminus X_v$. But now S' is a maximal stable set in $G \setminus X_v$ with $S' \cap K = \emptyset$, contrary to the fact that K is a dominant clique in $G \setminus X_v$. This proves that K' is a dominant clique and therefore G is resolved. So we may assume that $G \setminus X_v$ is not perfect and there exists $k \in \{2, 3\}$ such that every maximal stable set in $G \setminus X_v$ has size k. It follows that G is not perfect. Since $G \setminus X_v$ and $G \setminus X_w$ are isomorphic and every maximal stable set in G is either contained in $V(G \setminus X_v)$ or in $V(G \setminus X_w)$, it follows that every maximal stable set in G has size k, and therefore G is resolved. This proves that every graphic thickening of T is resolved and, thus, T

is resolved, completing the proof of (5.2.13). ■

5.3 Proofs of Theorem 5.0.6 and Theorem 5.0.7

In this section, we give the proofs of Theorem 5.0.6 and Theorem 5.0.7.

5.3.1 Fractionally co-strongly perfect graphs are $\mathcal{F}$-free

We first prove a result on saturating vertex weightings in graphs that display a certain symmetry:

(5.3.1) *Let G be a graph that has a saturating vertex weighting. Let $\phi : V(G) \to V(G)$ be an automorphism for G. Then there exists a saturating vertex weighting $\bar{w}$ such that $\bar{w}(x) = \bar{w}(\phi(x))$ for every $x \in V(G)$.*

Proof. Suppose that w is a saturating vertex weighting for G. Let $\phi^1 = \phi$ and for $k \geq 2$, let $\phi^k = \phi^{k-1} \circ \phi$. Since a set $S \subseteq V(G)$ is stable if and only if $\phi^k(S)$ is stable, it follows that $w \circ \phi^k$ is a saturating vertex weighting for G. Let $K \geq 1$ be such that $V(G) = \phi^K(V(G))$ and consider the function $\bar{w} = \frac{1}{K}\sum_{i=0}^{K-1} w \circ \phi^i$. Since $\bar{w}$ is a convex combination of solutions to the system of linear equations (5.1), it follows that $\bar{w}$ is a solution to (5.1) and, therefore, $\bar{w}$ is a saturating vertex weighting. Now observe that $\bar{w} = \bar{w} \circ \phi$. This proves (5.3.1). ■

Next, we need the following technical result. For a connected graph G, we say that $X \subseteq V(G)$ is a *clique cutset* if X is a clique and $G \setminus X$ is disconnected.

(5.3.2) *Let G be a graph, let X be a clique cutset in G, let B be a connected component of $G \setminus X$ and let $G' = G \setminus V(B)$. Suppose that for every $x \in X$, $G|(V(B) \cup \{x\})$ is a heft with end x and $N(x) \cap V(B) = N(x') \cap V(B)$ for all $x, x' \in X$. Suppose in addition that there exists a maximal stable set in G' that does not meet X. Then, every saturating vertex weighting w for G satisfies $w(v) = 0$ for all $v \in V(B)$.*

Proof. Let $x \in X$, $i \in [3]$, and $k \geq 0$ be such that $B' = G|(V(B) \cup \{x\})$ is isomorphic to the heft $\mathcal{H}_i(k)$. We prove the lemma for the case when $i = 1$ only, as the other two cases are analogous. Let $P = p_1\text{-}p_2\text{-}\dots\text{-}p_k = x$ be the rope of B' and let $v_1, v_2, \dots, v_5$ be the other vertices of B', labeled as in Figure 5.1(b). We use induction on k. Let w be a saturating vertex weighting for G. By (5.3.1), we may assume that $w(v_2) = w(v_5)$ and $w(v_3) = w(v_4)$. First suppose that $k = 0$. Let S be a maximal stable set in G' such that $x \in S$. Let $S_1 = S \cup \{v_2, v_5\}$ and let $S_2 = S \cup \{v_1\}$. Since w is a saturating vertex weighting and S_1 and S_2 are maximal stable sets with $S_1 \setminus S_2 = \{v_2, v_5\}$ and $S_2 \setminus S_1 = \{v_1\}$, it follows that $w(v_1) = w(v_2) + w(v_5) = 2w(v_5)$. Now let S' be a maximal stable set in G such that $v_3 \in S$. Clearly, either $v_1 \in S'$ or $v_5 \in S'$. Let $S_1' = (S' \setminus \{v_1\}) \cup \{v_5\}$ and $S_2' = (S' \setminus \{v_5\}) \cup \{v_1\}$. Since w is a saturating vertex weighting and S_1' and S_2' are maximal stable sets with $S_1' \setminus S_2' = \{v_5\}$ and $S_2' \setminus S_1' = \{v_1\}$, it follows that $w(v_5) = w(v_1)$. Combining this with the equality found above, it follows that $w(v_1) = 2w(v_1)$ and hence that $w(v_1) = w(v_2) = w(v_5) = 0$. Finally, let S'' be a maximal stable set in G' such that S'' does not meet X. Let $S_1'' = S'' \cup \{v_3, v_5\}$ and $S_2'' = S'' \cup \{v_2, v_5\}$. Since w is a saturating vertex weighting and S_1'' and S_2'' are maximal stable sets with $S_1'' \setminus S_2'' = \{v_3\}$ and $S_2'' \setminus S_1'' = \{v_2\}$, it follows that $w(v_3) = w(v_2) = 0$ and, hence, $w(v_4) = 0$. This proves the claim for $k = 0$.

Next, suppose that $k \geq 1$ and let y be the unique neighbor of x in $V(B)$. Since $\{y\}$ is a clique cutset, B is isomorphic to the heft $\mathcal{H}_1(k-1)$, and there clearly exists a maximal stable set in $G|(V(G') \cup \{y\})$ that does not meet y. Now it follows from the induction hypothesis that $w(v) = 0$ for all $v \in V(B) \setminus \{y\}$ and therefore it suffices to show that $w(y) = 0$. Let S be a maximal stable set in G' such that $S \cap X = \emptyset$. Let S_1 be a maximal stable set in B such that $y \in S_1$ and let S_2 be a maximal stable set in B such that $y \notin S_2$. Since $S \cup S_1$ and $S \cup S_2$ are maximal stable sets, it follows that

$$\sum_{v \in S_2} w(v) = \sum_{v \in S_1} w(v) = w(y) + \sum_{v \in S_1 \setminus \{y\}} w(v).$$

Now, since $\sum_{v \in S_1 \setminus \{y\}} w(v) = \sum_{v \in S_2} w(v) = 0$, it follows that $w(y) = 0$. This proves (5.3.2). ■

This puts us in a position to prove Theorem 5.0.6, the statement of which we repeat for clarity:

Theorem 5.0.6. *Let G be a fractionally co-strongly perfect graph. Then G is $\mathcal{F}$-free.*

Proof. It suffices to show that no graph in $\mathcal{F}$ is fractionally co-strongly perfect. First, let $H \in \mathcal{F}_1$, *i.e.*, H is a cycle of length $n = 2k$, $k \geq 3$ or of length $n = 2k+1, k \geq 4$. Suppose that there exists a saturating vertex weighting w for H. It follows from (5.3.1) that there exists $c \in [0,1]$ such that $w : V(H) \to [0,1]$ with $w(v) = c$ for every $v \in V(H)$. Let $v_1, v_2, \dots, v_n$ be the vertices of H in order. Since $\{v_2, v_4, \dots, v_{2k}\}$ is a maximal stable set of cardinality k, it follows that $c = \frac{1}{k}$. Now let $S = \{v_1, v_4, v_6, \dots, v_{2(k-1)}\}$. S is a maximal stable set, but $\sum_{v \in S} w(v) = \frac{k-1}{k} < 1$, a contradiction.

Suppose that there exists a saturating vertex weighting w for $\mathcal{G}_1$. It follows from (5.3.1) and the fact that the graph is symmetric along the vertical axis that we may assume that $w(v_2) = w(v_5)$, $w(v_3) = w(v_4)$ and $w(v_6) = w(v_7)$. Let $y = w(v_2) = w(v_5)$. By looking at different stable sets, we obtain the following equations:

$$\begin{array}{lcl}
\{v_3, v_5\}, \{v_5, v_6\} & \implies & w(v_3) = w(v_4) = w(v_6) = w(v_7) = 1 - y; \\
\{v_1, v_3\} & \implies & w(v_1) = 1 - w(v_3) = y \\
\{v_1, v_6, v_7\} & \implies & y + 2(1-y) = 1 \Leftrightarrow y = 1.
\end{array}$$

But now consider the maximal stable set $\{v_2, v_5\}$. It satisfies $w(v_2) + w(v_5) = 2y = 2$, a contradiction. This proves that $\mathcal{G}_1$ is not fractionally co-strongly perfect. The proofs for the graphs $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ are analogous.

Finally, consider any $H \in \mathcal{F}_3$. It follows that H is a skipping rope. Let (H_1, k_1), (H_2, k_2), x_1, and x_2 be as in the definition of a skipping rope. By applying (5.3.2) to each of the two hefts, it follows that every saturating vertex weighting w satisfies $w(v) = 0$ for all $v \in V(H)$, clearly contradicting the fact such w is a saturating vertex weighting. Hence, H has no saturating vertex weighting and therefore H is not fractionally co-strongly perfect. This proves Theorem 5.0.6. ■

5.3.2 Perfectly resolved claw-free graphs are fractionally co-strongly perfect

The next step is to show that perfectly resolved graphs are fractionally co-strongly perfect. We start with a simple lemma:

(5.3.3) *A graph G is fractionally co-strongly perfect if and only if every connected component of G is fractionally co-strongly perfect.*

Proof. The 'only-if' direction follows immediately from the definition of fractional co-strongly perfection. For the 'if' direction, let H be an induced subgraph of G. Let $C_1, C_2, \dots, C_q$ be the connected components of G and, for $i \in [q]$, let $H_i = G|(V(H) \cap V(C_i))$. From the symmetry, we may assume that $V(H_1) \neq \emptyset$. Since C_1 is fractionally co-strongly perfect, so is H_1 and, hence, there exists $w_1 : V(H_1) \to [0,1]$ such that $\sum_{v \in T} w_1(v) = 1$ for every maximal stable set T of H_1. Now define $w : V(H) \to [0,1]$ by $w(u) = w_1(u)$ for all $u \in V(H_1)$ and $w(v) = 0$ for all $v \in V(H) \setminus V(H_1)$. Let S be a maximal stable set S of H. Since $S \cap V(H_1) \neq \emptyset$ is a maximal stable set in H_1, it follows that $\sum_{v \in S} w(v) = \sum_{v \in S \cap V(H_1)} w(v) = 1$, thus proving (5.3.3). ■

This lemma enables us to prove Theorem 5.0.7, the statement of which we repeat for clarity:

Theorem 5.0.7. *Let G be a claw-free graph. If G is perfectly resolved, then G is fractionally co-strongly perfect.*

Proof. Let G' be an induced subgraph of G. We argue by induction on $|V(G')|$. It follows from (5.3.3) that we may assume that G' is connected. It suffices to show that G' has a saturating vertex weighting. Since G is perfectly resolved, G' is resolved. It follows that either there exists $x \in V(G')$ such that x is complete to $V(G') \setminus \{x\}$, or G' has a dominant clique, or G' is not perfect and there exists $k \in \{2,3\}$ such that every maximal stable set in G' has size k. First, suppose that there exists $x \in V(G')$ such that x is complete to $V(G') \setminus \{x\}$. It follows from the inductive hypothesis that $G' \setminus \{x\}$ has a saturating vertex weighting w_0. Define $w : V(G') \to [0,1]$ by setting $w(x) = 1$ and $w(v) = w_0(v)$ for all $v \in V(G') \setminus \{x\}$. It is not hard to see that this is a saturating vertex weighting for G' and the claim holds. Next, suppose that G' has a dominant clique K. Define $w : V(G') \to [0,1]$ by $w(v) = 1$ if $v \in K$ and $w(v) = 0$ otherwise. This is clearly a saturating vertex weighting for G' and, hence, the claim holds. Finally, suppose that G' is not perfect and there exists k such that every maximal stable set in G' has cardinality k. Now $w : V(G') \to [0,1]$ defined by $w(v) = 1/k$ for all $v \in V(G')$ is clearly a saturating vertex weighting for G'. Therefore, the claim holds. This proves Theorem 5.0.7. ■

5.4 Theorem 5.0.8 for $\mathcal{F}$-free basic claw-free graphs

In this section, the goal is to prove Theorem 5.0.8 using the structure theorem for claw-free graphs, Theorem 5.2.3. In fact, we prove the following:

(5.4.1) *Every $\mathcal{F}$-free basic claw-free trigraph is resolved.*

Since an $\mathcal{F}$-free claw-free trigraph T is resolved if and only if every $\mathcal{F}$-free graphic thickening of T is resolved, Theorem 5.0.8 is an immediate corollary of (5.4.1). We prove (5.4.1) by dealing with the outcomes of Theorem 5.2.3 separately. We first make the following easy observation concerning trigraphs from the icosahedron:

(5.4.2) *No trigraph from the icosahedron is $\mathcal{F}$-free.*

Proof. Let T be a trigraph from the icosahedron and let $v_1, v_2, \dots, v_9$ be as in the definition of T. Then, v_1-v_3-v_5-v_6-v_8-v_9-v_1 is a weakly induced cycle of length six in T, and thus T is not $\mathcal{F}$-free. This proves (5.4.2). ■

We will deal with the remaining outcomes of (5.4.1), namely antiprismatic trigraphs, circular interval trigraphs, and trigraphs that are the union of three cliques, in Subsections 5.4.1, 5.4.2, and 5.4.3, respectively.

5.4.1 $\mathcal{F}$-free antiprismatic trigraphs

The following lemma deals with $\mathcal{F}$-free antiprismatic trigraphs.

(5.4.3) *Every $\mathcal{F}$-free antiprismatic trigraph is resolved.*

Proof. Let T be an $\mathcal{F}$-free antiprismatic trigraph. If T contains no triad, then T is resolved by (5.2.10). Thus, we may assume that T contains a triad $\{a_1, a_2, a_3\}$. Let B_1 be the vertices that are complete to $\{a_2, a_3\}$, B_2 the vertices that are complete to $\{a_1, a_3\}$, and B_3 the vertices that are complete to $\{a_1, a_2\}$. Since T is antiprismatic, it follows that $V(T) = \{a_1, a_2, a_3\} \cup B_1 \cup B_2 \cup B_3$. We may assume that T is not resolved. We give the proof using a number of claims.

(i) *For distinct $i, j \in [3]$, a_i is strongly antiadjacent to a_j and $B_i \cup B_j$ is not a strong clique.*

We may assume that $i = 1$, $j = 2$. First suppose that $a_1a_2 \in F(T)$. If $b_1, b_1' \in B_1$ are antiadjacent, then a_2 is complete to the triad $\{a_1, b_1, b_1'\}$, contrary to (5.2.2). Thus, B_1 is a strong clique and, by the symmetry, B_2 is a strong clique. If B_1 is strongly complete to B_2, then a_3 is a simplicial vertex, contrary to (5.2.9). Thus, there exist antiadjacent $b_1 \in B_1$ and $b_2 \in B_2$. But now, (5.2.12) applied to a_1-a_2-b_1-a_3-b_2-a_1 implies that T is resolved, a contradiction. This proves that $a_1a_2 \notin F(T)$, and thus a_1 is strongly antiadjacent to a_2. Now suppose that $B_1 \cup B_2$ is a strong clique. Then, a_3 is a simplicial vertex, contrary to (5.2.9). This proves **(i)**. $\square$

(ii) *Let $i, j \in [3]$ be distinct. Let $x_1, x_2 \in B_i$ be antiadjacent. Then, B_j can be partitioned into sets $B_j(x_1)$, $B_j(x_2)$ such that, for $\{k, l\} = \{1, 2\}$, x_k is strongly complete to $B_j(x_k)$ and strongly anticomplete to $B_j(x_l)$.*

From the symmetry, we may assume that $i = 1$ and $j = 2$. If x_1 and x_2 have a common neighbor $z \in B_2$, then z is complete to the triad $\{a_1, x_1, x_2\}$, a contradiction. If x_1 and x_2 have a common antineighbor $z' \in B_2$, then a_3 is complete to the triad $\{x_1, x_2, z'\}$, a contradiction. Thus, x_1 and x_2 have no common neighbor and no common antineighbor in B_2. It follows that for every $z \in B_2$, one of x_1, x_2 is strongly adjacent to z, and the other is strongly antiadjacent to z. This proves **(ii)**. $\square$

(iii) *There is no triad $\{b_1, b_2, b_3\}$ with $b_i \in B_i$ for $i = 1, 2, 3$.*

Suppose that $\{b_1, b_2, b_3\}$ is a triad with $b_i \in B_i$. Then a_1-b_3-a_2-b_1-a_3-b_2-a_1 is a weakly induced cycle of length six, a contradiction. This proves **(iii)**. $\square$

(iv) *B_1, B_2, B_3 are all nonempty strong cliques.*

First suppose for a contradiction that, for $i = 1, 2$, there exist antiadjacent $p_i, q_i \in B_i$. It follows from **(ii)** that we may assume that p_1 is strongly adjacent to p_2 and strongly antiadjacent to q_2, and q_1 is strongly adjacent to q_2 and strongly antiadjacent to p_2. Now, a_2-p_1-p_2-a_1-q_2-q_1-a_2 is a weakly induced cycle of length six, a contradiction. This proves that at most one of B_1, B_2, B_3 is not a strong clique.

Next, suppose that $B_1 = \emptyset$. Since at most one of B_2, B_3 is not a strong clique, we may assume that B_2 is a strong clique. But now $B_1 \cup B_2$ is a strong clique, contrary to **(i)**. This proves that B_1, B_2 and B_3 are all nonempty.

We may assume that B_1 is not a strong clique, because otherwise the claim holds. It follows that B_2 and B_3 are strong cliques. Let $x, y \in B_1$ be antiadjacent. For $i = 2, 3$, let $B_i(x) \subseteq B_i$ and $B_i(y) \subseteq B_i$ be as in **(ii)** applied to x, y, B_1, and B_i. It follows from **(iii)** that $B_2(x)$ is strongly complete to $B_3(x)$ and $B_2(y)$ is strongly complete to $B_3(y)$. Hence, from **(i)** and the symmetry, we may assume that there exist antiadjacent $x_2 \in B_2(x)$ and $y_3 \in B_3(y)$. If there exists $x_3 \in B_3(x)$, then $T|\{x_2, x_3, y_3, y, a_3, x, a_1\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. This proves that $B_3(x) = \emptyset$ and, by the symmetry, that $B_2(y) = \emptyset$. Observe that this implies that $B_2(x) = B_2$ and $B_3(y) = B_3$.

So we may assume that for every two antiadjacent $x', y' \in B_1$, one of x', y' is strongly complete to B_2 and strongly anticomplete to B_3, and the other is strongly complete to B_3 and strongly anticomplete to B_2. Since $B_2, B_3 \neq \emptyset$, it follows that the complement of $T|B_1$ contains no odd cycles, and thus B_1 is the union of two strong cliques. For $i = 2, 3$, let $Z_i \subseteq B_1$ be the set of vertices in B_1 that have an antineighbor in B_1 and that are strongly complete to B_i. It follows that Z_2 and Z_3 are strong cliques. Let $Z^* = B_1 \setminus (Z_2 \cup Z_3)$. By definition, Z^* is a strong clique and Z^* is strongly complete to $Z_1 \cup Z_2$.

Now observe (Z_2, Z_3) is a homogeneous pair of strong cliques. It follows from **(i)** that there exist antiadjacent $b_2 \in B_2$ and $b_3 \in B_3$. But now, by (5.2.11) applied to (Z_2, Z_3) and the weakly induced path b_2-a_1-b_3, it follows that T is resolved, a contradiction. This proves **(iv)**. □

(v) *Let $\{i, j, k\} = \{1, 2, 3\}$. Let $b_i \in B_i$ and $b_j \in B_j$ be antiadjacent. Then, at least one of b_i, b_j is strongly complete to B_k.*

We may assume that $i = 1$, $j = 2$, $k = 3$. Suppose that b_1 has an antineighbor $x \in B_3$ and b_2 has a antineighbor $y \in B_3$. It follows from **(iii)** that $x \neq y$ and that x is strongly adjacent to b_2 and y is strongly adjacent to b_1. It follows from **(iv)** that x is strongly adjacent to y. Now, $T|\{a_3, b_1, x, y, b_2, a_1, a_2\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. This proves **(v)**. □

(vi) *Let $\{i, j, k\} = \{1, 2, 3\}$. Then, no vertex in B_i has antineighbors in both B_j and B_k.*

We may assume that $i = 1$, $j = 2$, $k = 3$. Suppose that $b_1 \in B_1$ has antineighbors $b_2 \in B_2$ and $b_3 \in B_3$. It follows from **(iii)** that b_2 is strongly adjacent to b_3. It follows from **(v)** that b_2 is strongly complete to B_3 and b_3 is strongly complete to B_2. From **(i)**, there exist antiadjacent $b_2' \in B_2$ and $b_3' \in B_3$. It follows that $\{b_2, b_3\} \cap \{b_2', b_3'\} = \emptyset$. It follows from **(v)** that b_2', b_3' are both strongly complete to B_1. Now $T|\{b_3, b_2, a_3, b_1, a_2, b_2', b_3', a_1\}$ contains $\mathcal{G}_3$ as a weakly induced subgraph, a contradiction. This proves **(vi)**. □

It follows from **(i)** that for $i = 1, 2, 3$, there exist $x_i, y_i \in B_i$ such that the pairs x_1y_2, x_2y_3, x_3y_1 are antiadjacent. It follows from **(iv)** and **(vi)** that $x_i \neq y_i$ for $i = 1, 2, 3$ and all pairs among $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ except the aforementioned are strongly adjacent. Now, $T|\{a_1, x_1, y_1, a_2, x_2, y_2, a_3, x_3, y_3\}$ contains $\mathcal{G}_4$ as a weakly induced subgraph, a contradiction. This proves (5.4.3). ■

5.4.2 $\mathcal{F}$-free long circular interval trigraphs

In this section, we prove that $\mathcal{F}$-free long circular interval trigraphs are resolved. Let T be a long circular interval graph and let $\Sigma, F_1, \dots, F_k$ be as in the definition of a long circular interval graph. If, in addition, $\bigcup_{i=1}^{k} F_i \neq \Sigma$, then T is called a *linear interval trigraph*. We start with the following easy result, which shows that we may assume that the long circular interval trigraphs that we are dealing with in this section are really long circular interval trigraphs and not linear interval trigraphs.

(5.4.4) *Every linear interval trigraph is resolved.*

Proof. Let T be a linear interval trigraph. Thus, we may order the vertices of T as $v_1, v_2, \dots, v_n$ such that for $i < j$, if v_i is adjacent to v_j, then v_k is strongly adjacent to v_l for all $i < k \leq l \leq j$. It follows that $N(v_1)$ is a strong clique and hence that v_1 is a simplicial vertex in T. Thus, T is resolved by

(5.2.9). This proves (5.4.4). ■

In handling long circular interval trigraphs, it turns out to be convenient to make a distinction depending on the existence of a semihole of length at least five in the trigraph. Section 5.4.2 deals with the case where the trigraph contains no semihole of length at least five. It will turn out that there are two types of such trigraphs, namely ones that have a structure that is similar to the complement of a 7-cycle, and ones that have a structure that is similar to a 4-cycle with certain attachments. Section 5.4.2 deals with the remaining case where the trigraph does contain such semihole. In this case, the trigraph has a structure that is similar to either a 5-cycle or a 7-cycle, with certain attachments.

Long circular interval trigraphs with no long semiholes

Let $\bar{C}_7$ be a graph that is the complement of a 7-cycle. We say that a trigraph T is *of the* $\bar{C}_7$ *type* if $V(T)$ can be partitioned into seven nonempty strong cliques $W_1, \dots, W_7$ such that for all $i \in [7]$, W_i is strongly complete to W_{i+1}, W_i is complete to W_{i+2}, W_i is strongly anticomplete to W_{i+3} (where subscript arithmetic is modulo 7). We first look at long circular interval trigraphs with no long semiholes that contain $\bar{C}_7$ as a weakly induced subgraph.

(5.4.5) *Let T be a long circular interval trigraph with no semihole of length at least five. If T contains $\bar{C}_7$ as a weakly induced subgraph, then T is of the $\bar{C}_7$ type.*

Proof. Let $W_1, W_2, \dots, W_7 \subseteq V(T)$ be such that for all $i \neq j$ (with subscript arithmetic modulo 7), W_i is a nonempty clique, $W_i \cap W_j = \emptyset$, W_i is complete to $W_{i+1} \cup W_{i+2}$, W_i is anticomplete to $W_{i+3} \cup W_{i+4}$, and $\bigcup_{i=1}^{7} W_i$ is maximal. The cliques W_i, $i \in [7]$, exist since T contains $\bar{C}_7$ as a weakly induced subgraph. We start with some claims:

(i) *For $i \in [7]$, W_i is strongly anticomplete to $W_{i+3} \cup W_{i+4}$.*

Without loss of generality we may assume that $i = 1$, and from the symmetry it follows that it is enough to show that W_1 is strongly anticomplete to W_4. So suppose that there exists a vertex $x \in W_1$ which is semiadjacent to some vertex $y \in W_4$. From the definition of a trigraph, it follows that x is strongly complete to $W_6 \cup W_7$ and strongly anticomplete to W_5, and y is strongly complete to $W_5 \cup W_6$ and strongly anticomplete to W_7. But now any vertex $z \in W_6$ is complete to the semihole $\{x, u, v, y\}$, where $u \in W_7$ and $v \in W_5$, which contradicts (5.2.5). □

(ii) *Suppose that x has a neighbor in W_i. Then,*

(a) *x is complete to at least one of W_{i-1}, W_{i+1}; and*

(b) *x is complete to at least one of W_{i-1}, W_{i+2}; and*

(c) *x is complete to at least one of W_{i-2}, W_{i+2}.*

Let y_i be a neighbor of x in W_i. Suppose that x has a strong antineighbor $y_{i-1} \in W_{i-1}$ and a strong antineighbor $y_{i+1} \in W_{i+1}$. If x has an antineighbor $y_{i-2} \in W_{i-2}$, then y_i is complete to the triad $\{x, y_{i+1}, y_{i-2}\}$, a contradiction. Thus x is complete to W_{i-2}. From the symmetry, it follows that x is complete to W_{i+2}. Let $y_{i-2} \in W_{i-2}$ and $y_{i+2} \in W_{i+2}$. Now x-y_{i-2}-y_{i-1}-y_{i+1}-y_{i+2}-x is a semihole of length five, a contradiction. This proves part (a). Next suppose that x has a strong antineighbor $y_{i-1} \in W_{i-1}$ and a strong antineighbor $y_{i+2} \in W_{i+2}$. Then y_i is complete to the triad $\{y_{i-1}, y_{i+2}, x\}$, a contradiction. This proves part (b). Finally suppose that x has a strong antineighbor $y_{i-2} \in W_{i-2}$ and a strong antineighbor $y_{i+2} \in W_{i+2}$. Then y_i is complete to the triad $\{y_{i-2}, y_{i+2}, x\}$, a contradiction. This proves part (c), thus completing the proof of **(ii)**. □

We claim that $V(T) = \bigcup_{i=1}^{7} W_i$. For suppose not. Then there exists $x \in V(T) \setminus \bigcup_{i=1}^{7} W_i$ with a neighbor in $\bigcup_{i=1}^{7} W_i$. Because $T|(\bigcup_{i=1}^{7} W_i)$ contains a semihole of length four, it follows from (5.2.5) that x has a neighbor in some set W_i. It follows from **(ii)** that, for some $i \in [7]$, x is complete to $W_i \cup W_{i+1}$. From the symmetry, we may assume that x is complete to $W_1 \cup W_2$. Now it follows from **(ii)** that x is complete to at least one of W_3, W_7. We may assume that x is complete to W_3. Finally, it follows from **(ii)** that x is complete to at least one of W_4, W_7. We may assume that x is complete to W_4. If x has a neighbor $y_6 \in W_6$, then let $y_1 \in W_1, y_2 \in W_2, y_4 \in W_4$ and observe that y_1-y_2-y_4-y_6-y_1 is a semihole of length four and x is complete to it, contrary to (5.2.5). This proves that x is strongly anticomplete to W_6,

(iii) *x is complete to exactly one of W_5, W_7 and strongly anticomplete to the other.*

Suppose that x has both a strong antineighbor $y_5 \in W_5$ and a strong antineighbor $y_7 \in W_7$. Then y_7-y_5-y_4-x-y_1-y_7, where $y_1 \in W_1$, is a semihole of length five, a contradiction. This proves that x is complete to one of W_5, W_7. Finally, suppose that x has a neighbor $y_5 \in W_5$ and a neighbor $y_7 \in W_7$. Let $y_2 \in W_2$ and $y_3 \in W_3$. Then x is a center for the semihole y_2-y_7-y_5-y_3-y_2, contrary to (5.2.5). This proves **(iii)**. □

From **(iii)**, we may assume that x is complete to W_5 and strongly anticomplete to $W_6 \cup W_7$. But now we may add x to W_3 and obtain a larger structure, a contradiction. This proves that $V(T) = \bigcup_{i=1}^{7} W_i$. The following claim states that many edges in $W_1 \cup W_2 \cup \cdots \cup W_7$ are in fact strong edges.

(iv) *For $i \in [7]$, $W_i \cup W_{i+1}$ is a strong clique.*

Suppose that $w, w' \in W_i \cup W_{i+1}$ are antiadjacent. Let $w_{i+2} \in W_{i+2}$ and $w_{i+4} \in W_{i+4}$. Then, w_{i+2} is anticomplete to the triad $\{w, w', w_{i+4}\}$, contrary to (5.2.2). This proves **(iv)**. □

It follows from the definition of $W_1, \dots, W_7$ and from **(iv)** that T is a trigraph of the $\bar{C}_7$ type. This proves (5.4.5). ■

The previous statement shows that if a long circular interval trigraph with no long semiholes contains $\bar{C}_7$ as a weakly induced subgraph, then it basically looks like $\bar{C}_7$. The following shows that such trigraphs have no triads, hence that they are resolved by (5.2.10):

(5.4.6) *Let T be a trigraph of the $\bar{C}_7$ type. Then T contains no triad.*

Proof. Let $W_1, W_2, \dots, W_7 \subseteq V(T)$ be as in the definition of a trigraph of the $\bar{C}_7$ type. Now suppose that T has a stable set $\{s_1, s_2, s_3\}$. Since W_i is a strong clique and W_i is strongly complete to W_{i+1}, it follows that for $j \neq k$, s_j and s_k are not in consecutive sets. Therefore, from the symmetry, we may assume that $s_1 \in W_1$, $s_2 \in W_3$, and $s_3 \in W_6$. It follows that s_1 is semiadjacent to both s_2 and s_3, a contradiction. This proves (5.4.6). ■

So, we may exclude $\bar{C}_7$ and concentrate on what happens otherwise. Let T be a trigraph. Let $A_1, A_2, A_3, A_4,\ B_1, B_2, B_3, B_4 \subseteq V(T)$ be strong cliques such that, for $i \in [4]$, (with subscript arithmetic modulo 4)

(1) if $i \in \{1, 3\}$, then A_i is complete to A_{i+1}, and if $i \in \{2, 4\}$, then A_i and A_{i+1} are linked, and

(2) A_i is strongly anticomplete to A_{i+2}, and

(3) B_i is strongly complete to $A_i \cup A_{i+1}$ and strongly anticomplete to $A_{i+2} \cup A_{i+3}$, and

(4) B_i is strongly anticomplete to B_j for $i \neq j$, and

(5) if $B_i \neq \emptyset$, then A_i is complete to A_{i+1}, and

(6) no vertex in A_i has antineighbors in both A_{i-1} and A_{i+1}.

We call such $(A_1, \dots, A_4, B_1, \dots, B_4)$ a *C_4-structure* in T. If, for T, there exists a C_4-structure $(A_1, \dots, A_4, B_1, \dots, B_4)$ such that $V(T) = A_1 \cup \dots \cup A_4 \cup B_1 \cup \dots \cup B_4$, then we say that T *admits a C_4-structure*. The following lemma states that if a long circular interval trigraph T with no long semiholes does not contain $\bar{C}_7$ as a weakly induced subgraph, then T is either a linear interval trigraph, or T admits a C_4-structure:

(5.4.7) *Let T be a long circular interval trigraph that has no semihole of length at least five. Then, either*

(1) *T is a linear interval trigraph, or*
(2) *T is of the $\bar{C}_7$ type, or*
(3) *T admits a C_4-structure.*

Proof. In view of outcomes (1) and (2), and (5.4.5), we may assume that T is not a linear interval trigraph and T has no weakly induced $\bar{C}_7$. This implies that T contains a weakly induced cycle of length four. Let $A_1, A_2, A_3, A_4 \subseteq V(T)$ be cliques in T such that:

(a) A_1 is complete to A_2 and A_3 is complete to A_4, and,

(b) A_1 is strongly anticomplete to A_3, and A_2 is anticomplete to A_4, and

(c) A_2 and A_3 are linked, and A_1 and A_4 are linked.

We may choose A_1, A_2, A_3, A_4 with maximal union. We call such quadruple a *structure*. Since T contains a weakly induced cycle of length four, it follows that $A_i \neq \emptyset$ for all $i \in [4]$. Let $A = \bigcup_{i=1}^{4} A_i$.

(i) *Let $v \in V(T) \setminus A$. Then there exists $i \in [4]$ such that v is strongly complete to $A_i \cup A_{i+1}$.*

Let u_1-u_2-u_3-u_4-u_1 with $u_i \in A_i$ be a semihole. Since T is claw-free, it follows from (5.2.5) that v is adjacent to at least two consecutive u_i's. Let k be such that v is adjacent to u_k and u_{k+1}. We may assume that $k \in \{1, 2\}$. First suppose that $k = 1$. Since no vertex is complete to u_1-u_2-u_3-u_4-u_1, we may assume that v is strongly antiadjacent to u_3. Since T is claw-free and u_2 is complete to A_1, it follows that v is strongly complete to A_1. If v is complete to A_4, then the claim holds, so we may assume that v has a strong antineighbor $a_4 \in A_4$. Let $a_1 \in A_1$ be a neighbor of a_4. Since a_1 is complete to A_2 and T is claw-free, it follows that v is strongly complete to A_2, as desired. So we may assume that $k = 2$ and v is strongly anticomplete to $\{u_1, u_4\}$. Suppose that v has an antineighbor $a_2 \in A_2$. Then a_2 is strongly antiadjacent to u_3, because otherwise u_3 is complete to the triad $\{u_4, v, a_2\}$, a contradiction. Since A_2 is linked to A_3, there exists a vertex $a_3 \in A_3$ such that a_2 is adjacent to a_3. Now a_3 is adjacent to u_2, because otherwise $T|\{u_1, a_2, u_2, u_4, a_3, u_3\}$ is a weakly induced $(1, 1, 1)$-prism, contrary to (5.2.5). This implies that v is adjacent to a_3, because otherwise u_2 is complete to the triad $\{u_1, v, a_3\}$. But now a_3 is complete to the triad $\{u_4, v, a_2\}$, a contradiction. Thus v is strongly complete to A_2, and from the symmetry v is also complete to A_3. This proves **(i)**. □

(ii) *Suppose that, for some $i \in [4]$, $v \in V(T) \setminus A$ is strongly complete to $A_i \cup A_{i+1}$. Then v is strongly anticomplete to $A_{i+2} \cup A_{i+3}$.*

From the symmetry, we may assume that $i \in \{1, 2\}$. For $j = i+2, i+3$, let $Z_j = N(v) \cap A_j$ and let $Y_j = A_j \setminus Z_j$.

First suppose that both Z_{i+2} and Z_{i+3} are nonempty. Because no vertex is complete to a semihole of length four by (5.2.5), it follows that Z_{i+2} is strongly anticomplete to Z_{i+3}. It follows that $i = 2$. Now let $x_4 \in Z_4$ and $x_1 \in Z_1$. Since A_4 and A_1 are linked, x_4 has a neighbor $y_1 \in Y_1$ and x_1 has a neighbor $y_4 \in Y_4$. Since x_4 is complete to $\{v, y_4, y_1\}$, the latter is not a triad and hence it follows that y_4 is adjacent to y_1. But now $T|\{x_4, u_2, y_4, v, y_1, u_3, x_1\}$ contains $\bar{C}_7$ as weakly induced subgraph, a contradiction.

So we may assume that at least one of Z_{i+2}, Z_{i+3} is empty. If both are empty, then v is strongly anticomplete to $A_{i+2} \cup A_{i+3}$ and the claim holds. Therefore, from the symmetry, we may assume that $Z_{i+2} \neq \emptyset$ and $Z_{i+3} = \emptyset$. If $i = 1$, then we may add v to A_2 and obtain a larger structure, a contradiction. If $i = 2$ and $Y_{i+2} = \emptyset$, then we may add v to A_3 and obtain a larger structure, a contradiction. Hence, we may assume that $i = 2$ and $Y_4 \neq \emptyset$.

Now suppose that $a_2 \in A_2$ and $a_3 \in A_3$ are strongly antiadjacent. Let $q_1 \in A_1$ and $y_4 \in Y_4$ be adjacent. Then a_2-v-a_3-y_4-q_1-a_2 is a semihole of length five, a contradiction. This proves that A_2 is complete to A_3.

We claim that for every $a_1 \in A_1$, $x_4 \in Z_4$ and $y_4 \in Y_4$, a_1 is either complete or strongly anticomplete to $\{x_4, y_4\}$. For suppose not. If a_1 is adjacent to x_4 and strongly antiadjacent to y_4, then x_4 is complete to the triad $\{a_1, y_4, v\}$, a contradiction. So we may assume that a_1 is adjacent to y_4 and strongly antiadjacent to x_4. But now, v-x_4-y_4-a_1-u_2-v is a semihole of length five, a contradiction. This proves the claim.

Since Z_4 and Y_4 are both nonempty, it follows that every vertex in A_1 is either complete or anticomplete to A_4. Since every vertex in A_1 has a neighbor in A_4, this implies that A_1 is complete to A_4. But now, letting $A'_1 = A_2$, $A'_2 = A_3 \cup \{v\}$, $A'_3 = A_4$ and $A'_4 = A_1$, we obtain a larger structure, a contradiction. This proves **(ii)**. □

For $i \in [4]$, let B_i be the vertices that are strongly complete to $A_i \cup A_{i+1}$. It follows from **(ii)** that B_i is strongly anticomplete to $A_{i+2} \cup A_{i+3}$. It follows from **(ii)** that $V(T) = A_1 \cup \ldots \cup A_4 \cup B_1 \cup \ldots \cup B_4$. The next few claims state some properties of the sets $A_1, \ldots, A_4, B_1, \ldots, B_4$.

(iii) *For $i \in [4]$, no vertex in A_i has both an antineighbor in A_{i+1} and an antineighbor in A_{i-1}.*

Suppose that $a_i \in A_i$ has nonneighbors $a_{i+1} \in A_{i+1}$ and $a_{i-1} \in A_{i-1}$. From the symmetry, we may assume that $i = 1$. Since A_1 is complete to A_2, it follows that a_1 and a_2 are semiadjacent and hence that a_1 and a_4 are strongly antiadjacent. Now let $a'_1 \in A_1$ be a neighbor of a_4. Since A_1 is complete to A_2, it follows that a'_1 is adjacent to a_2. Now a'_1 is complete to the triad $\{a_1, a_2, a_4\}$, a contradiction. This proves **(iii)**. □

(iv) *For $i, j \in [4]$, B_i is strongly anticomplete to B_j for $j \neq i$.*

Let $i \in \{1, 3\}$. If $b_i \in B_i$ is adjacent to $b_{i+1} \in B_{i+1}$, then b_i-b_{i+1}-u_{i+2}-u_{i+3}-u_i-b_i is a semihole of length five, a contradiction. If $b_i \in B_i$ is adjacent to $b_{i+2} \in B_{i+2}$, then $T|\{u_i, u_{i+1}, u_{i+2}, u_{i+3}, b_i, b_{i+1}\}$ contains a weakly induced $(1,1,1)$-prism, contrary to (5.2.5). Thus, it follows from the symmetry that B_i is strongly anticomplete to B_j for $j \neq i$. This proves **(iv)**. □

(v) *For $i \in [4]$, if $B_i \neq \emptyset$, then A_i is complete to A_{i+1}.*

This is trivial if $i = 1, 3$. So from the symmetry we may assume that $i = 2$. If $a_2 \in A_2$ and $a_3 \in A_3$ are nonadjacent, then for any vertex $b_2 \in B_2$, a_2-b_2-a_3-u_4-u_1-a_2 is a semihole of length five, a contradiction. This proves **(v)**. □

We claim that T admits a C_4-structure. We already noted that $A_1, \ldots, A_4, B_1, \ldots, B_4$ is a partition of $V(T)$. Properties (1)-(6) in the definition of a C_4-structure follow from the definition of $A_1, \ldots, A_4, B_1, \ldots, B_4$ and **(iii)**, **(iv)**, and **(v)**. This proves (5.4.7). ■

We are now ready to prove the first main result of this subsection.

(5.4.8) *Every $\mathcal{F}$-free long circular interval trigraph with no semihole of length at least five is resolved.*

Proof. Let T be long circular interval trigraph with no semihole of length at least five. It follows from (5.4.7) that either T is a linear interval trigraph, or T is of the $\bar{C}_7$ type, or T admits a C_4-structure. If T is a linear interval trigraph, then the lemma holds by (5.4.4). If T is of the $\bar{C}_7$ type, then the lemma holds by (5.4.6). Therefore, we may assume that T admits a C_4-structure. Let $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ be as in the definition of a C_4-structure. We may assume that T is not resolved.

(i) *If, for some $i \in [4]$, $B_i \neq \emptyset$, then A_i is not strongly complete to A_{i+1}.*

Suppose that $B_i \neq \emptyset$ and A_i is strongly complete to A_{i+1}. Then any vertex in B_i is a simplicial vertex and hence T is resolved by (5.2.9), a contradiction. This proves **(i)**. □

(ii) *If, for some $i \in [4]$, $B_i \neq \emptyset$, then A_{i+2} is strongly complete to A_{i+3}.*

Let i be such that $B_i \neq \emptyset$ and suppose that there exist two antiadjacent vertices $x \in A_{i+2}$ and $y \in A_{i+3}$. It follows from **(i)** that there exist antiadjacent $a_i \in A_i$ and $a_{i+1} \in A_{i+1}$. If x is semiadjacent to y, then a_i-b_i-a_{i+1}-x-y-a_i is a weakly induced cycle of length five and $xy \in F(T)$ and, thus, T is resolved by (5.2.12), a contradiction. Thus, x is strongly antiadjacent to y. It follows from property (6) of a C_4 structure that a_i is strongly complete to A_{i+3} and a_{i+1} is strongly complete to A_{i+2}. Now let $x' \in A_{i+2}$ be a neighbor of y and let $y' \in A_{i+3}$ be a neighbor of x. If x' and y' are adjacent, then $T|\{b_i, a_{i+1}, x', y', a_i, x, y\}$ contains $\mathcal{G}_1$ as weakly induced subgraph, a contradiction. Thus x' and y' are strongly antiadjacent.

We claim that no vertex in A_{i+3} is complete to $\{x, x'\}$. For suppose that such vertex $z \in A_{i+3}$ exists. Then, $T|\{b_i, a_{i+1}, x', z, a_i, x, y\}$ contains $\mathcal{G}_1$ as weakly induced subgraph, a contradiction. Hence, no vertex in A_{i+3} is complete to $\{x, x'\}$ and, in particular, every vertex in A_{i+3} has an antineighbor in A_{i+2}. Thus, property (6) of a C_4 structure implies that A_{i+3} is strongly complete to A_i. By the symmetry, A_{i+2} is strongly complete to A_{i+1}. But now, (A_{i+2}, A_{i+3}) is a homogeneous pair of cliques and a_i-b_i-a_{i+1} is a weakly induced path between their respective neighborhoods, and hence T is resolved by (5.2.11). This proves **(ii)**. □

(iii) *For each $i \in [4]$, at least one of B_i, B_{i+1} is empty.*

Suppose that for some $i \in [4]$, B_i and B_{i+1} are both nonempty. By **(i)**, there exist antiadjacent $a_i \in A_i$ and $a_{i+1} \in A_{i+1}$ and antiadjacent $a'_{i+1} \in A_{i+1}$ and $a'_{i+2} \in A_{i+2}$. It follows from property (6) of a C_4 structure that $a_{i+1} \neq a'_{i+1}$ and in particular a_{i+1} is strongly adjacent to a'_{i+2} and a'_{i+1} is strongly adjacent to a_i. Let $a_{i+3} \in A_{i+3}$ be a common strong neighbor of a_i and a_{i+2}. Such a_{i+3} exists since from **(ii)** it follows that A_{i+2} is strongly complete to A_{i+3} and A_{i+3} is strongly complete to A_i. But now $T|\{a_{i+3}, a_i, a'_{i+1}, a_{i+1}, a'_{i+2}, b_i, b_{i+1}\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. This proves **(iii)**. □

First suppose that $B_i = \emptyset$ for all $i \in [4]$. Then, it follows from property (6) of a C_4 structure that T does not contain a triad and, thus, T is resolved by (5.2.10). Hence, we may assume that $B_i \neq \emptyset$ for some $i \in [4]$. From **(i)**, **(ii)**, and **(iii)**, it follows that $B_j = \emptyset$ for all $j \neq i$. It follows from **(ii)** that A_{i+2} is complete to A_{i+3}. But now, T has no triad and hence T is resolved by (5.2.10). This proves

(5.4.8). ■

Long circular interval trigraphs with long semiholes

Lemma (5.4.8) deals with long circular interval trigraphs with no long semiholes. The following lemmas deal with the remaining case. The first lemma is an attachment lemma that describes how vertices can attach to a semihole in a long circular interval trigraph. We need some more definitions first. Let T be a trigraph and let C be a semihole of length k in T. Suppose that the vertices of C are ordered, so that $C = c_1\text{-}c_2\text{-}\dots\text{-}c_k\text{-}c_1$. Let $x \in V(T) \setminus V(C)$. Let $i \in [k]$. We say that x is a *hat of type i for C* if x is strongly complete to $\{c_i, c_{i+1}\}$ and strongly anticomplete to $V(C) \setminus \{c_i, c_{i+1}\}$. We say that x is a *clone of type i for C* if x is complete to $\{c_{i-1}, c_{i+1}\}$, strongly adjacent to c_i, and strongly anticomplete to $V(C) \setminus \{c_{i-1}, c_i, c_{i+1}\}$. Finally, we say that x is a *star of type i for C* if x is strongly antiadjacent to c_i and complete to $\{c_{i-1}, c_{i+1}\}$, and strongly complete to $V(C) \setminus \{c_{i-1}, c_i, c_{i+1}\}$.

(5.4.9) *Let T be an $\mathcal{F}$-free long circular interval trigraph. Let C be a semihole of length $k \geq 5$. Then, $k \in \{5, 7\}$, and every $x \in V(T) \setminus V(C)$ is either a hat, or a clone, or a star of type i for C, for some $i \in [k]$. Moreover, if x is a star for C, then $k = 5$.*

Proof. Let $C = c_1\text{-}c_2\text{-}\dots\text{-}c_k\text{-}c_1$. Since T is $\mathcal{F}$-free it follows that $k \in \{5, 7\}$. We first observe that:

($*$) if x is adjacent to c_i, then x is strongly adjacent to at least one of c_{i-1}, c_{i+1}, because otherwise $\{x, c_{i-1}, c_{i+1}\}$ is a triad and c_i is complete it.

It follows from (5.2.5) that C is dominating and has no center, and therefore x has at least one neighbor and one strong antineighbor in $V(C)$. We may assume that x is adjacent to c_1 and strongly antiadjacent to c_2. It follows from ($*$) that x is strongly adjacent to c_k. First suppose that x is adjacent to c_3. Then, by ($*$), x is strongly adjacent to c_4. If $k = 5$, then, x is a star of type 2, and the claim holds. So we may assume that $k = 7$. x is strongly antiadjacent to c_5 because otherwise x is complete to the triad $\{c_1, c_3, c_5\}$. Thus, by the symmetry, x is strongly antiadjacent to c_6. But now $C' = x\text{-}c_4\text{-}c_5\text{-}c_6\text{-}c_7\text{-}x$ is a semihole and c_2 has no neighbors in $V(C')$, contrary to (5.2.5). So we may assume that x is strongly antiadjacent to c_3. If $k = 5$, then, x is a clone of type 5 if x is adjacent to c_4 and x is a hat of type 5 if x is strongly antiadjacent to c_4. Thus we may assume that $k = 7$. Suppose that x is adjacent to c_4. x is strongly antiadjacent to c_6, because otherwise x is complete to the triad $\{c_1, c_4, c_6\}$, a contradiction. But now $c_1\text{-}c_2\text{-}c_3\text{-}c_4\text{-}x\text{-}c_1$ is a nondominating semihole and c_6 has no neighbor in it, contrary to (5.2.5). This proves that x is strongly antiadjacent to c_4. If x is adjacent to c_5, then $c_1\text{-}c_2\text{-}c_3\text{-}c_4\text{-}c_5\text{-}x\text{-}c_1$ is a weakly induced cycle of length six, a contradiction. Therefore, x is strongly antiadjacent to c_5. Now, x is a clone of type 7 if x is adjacent to c_6 and x is a hat of type 7 if x is strongly antiadjacent to c_6. This proves (5.4.9). ■

Next, we have two lemmas that describe the structure of an $\mathcal{F}$-free long circular interval trigraph that contains a semihole of length five and seven, respectively.

(5.4.10) *Let T be an $\mathcal{F}$-free long circular interval trigraph. Assume that T has a semihole of length five and no semihole of length seven. Then, $V(T)$ can be partitioned into 15 strong cliques $C_1, \dots, C_5$, $Y_1, \dots, Y_5$, $Z_1, \dots, Z_5$ such that for all $i, j \in [5]$, (subscript arithmetic is modulo 5)*

(1-a) *C_i is complete to C_{i+1} and strongly anticomplete to C_j with $j \notin \{i-1, i, i+1\}$,*

(1-b) *Y_i is strongly complete to $C_i \cup C_{i+1}$ and strongly anticomplete to C_j with $j \notin \{i, i+1\}$.*

(1-c) *Z_i is strongly complete to $C_{i+2} \cup C_{i+3}$, strongly anticomplete to C_i, and every vertex in Z_i is strongly complete to one of C_{i+1}, C_{i+4} and has a neighbor in the other,*

(1-d) *if $i \neq j$, then Y_i is strongly anticomplete to Y_j.*

Moreover, if there exists $y \in Y_i$, then:

(2-a) *Y_i is strongly complete to $Z_{i+2} \cup Z_{i+4}$, and strongly anticomplete to $Z_i \cup Z_{i+1} \cup Z_{i+3}$,*

(2-b) *$C_i \cup C_{i+1} \cup Z_{i+2} \cup Z_{i+4}$ is a strong clique.*

Proof. Let $C_1, \dots, C_5$ be cliques that satisfy property (1-a), and let $C = \bigcup_{i=1}^{5} C_i$ be maximal. Let $Y_1, \dots, Y_5 \subseteq V(T) \setminus C$ be cliques that satisfy property (1-b), and let $Y = \bigcup_{i=1}^{5} Y_i$ be maximal. Let $Z_1, \dots, Z_5 \subseteq V(T) \setminus (C \cup Y)$ be cliques that satisfy property (1-c), and let $Z = \bigcup_{i=1}^{5} Z_i$ be maximal. It follows from the fact that T has a semihole of length five that $C_i \neq \emptyset$ for $i \in [5]$. Furthermore, since T is claw-free it follows that each C_i, Y_i, and Z_i is a strong clique.

We claim that $V(T) = C \cup Y \cup Z$. So suppose for a contradiction that there exists $x \in V(T) \setminus (C \cup Y \cup Z)$. In what follows, we say that $F = f_1\text{-}f_2\text{-}\dots\text{-}f_5\text{-}f_1$ is an *aligned semihole in C* if $f_i \in C_i$ for all $i \in [5]$. It follows from (5.4.9) that, for every aligned semihole in C, x is either a star, a clone, or a hat. First suppose that x is star of type i, say, for some aligned semihole $F = f_1\text{-}f_2\text{-}\dots\text{-}f_5\text{-}f_1$ in C. From the symmetry, we may assume that $i = 1$. By rerouting F, it follows from the fact that T is claw-free that x is strongly complete to $C_3 \cup C_4$, and from (5.4.9) that x is strongly anticomplete to C_1. We claim that x is strongly complete to at least one of C_2, C_5. For suppose that x has antineighbors $c_2 \in C_2$ and $c_5 \in C_5$. Then, $T|(V(F) \cup \{c_2, c_5, x\})$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. By the maximality of Z_i, this means that $x \in Z_i$, a contradiction. So we may assume that x is not a star for any aligned semihole in C. Next, suppose that x is a clone of type i, say, for some aligned semihole $F = f_1\text{-}f_2\text{-}\dots\text{-}f_5\text{-}f_1$ in C. From the symmetry, we may assume that $i = 1$. By rerouting F, it follows from the fact that T is claw-free that x is strongly complete to C_1, and from (5.4.9) that x is strongly anticomplete to $C_3 \cup C_4$. We claim that x is complete to C_2. For suppose that x has a strong antineighbor $c_2' \in C_2$. Then, $c_2' \neq f_2$ and $T|(V(F) \cup \{c_2', x\})$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. Thus, x is complete to C_2 and, from the symmetry, to C_5. But now, by the maximality of C_1, $x \in C_1$, a contradiction. So we may assume that x is not a clone for any aligned semihole in C. It follows that x is a hat for every aligned semihole in C. Choose any aligned semihole $F = f_1\text{-}f_2\text{-}\dots\text{-}f_5\text{-}f_1$ in C. We may assume that x is a hat of type 1 for C. By rerouting F, it follows from (5.4.9) that x is strongly complete to $C_1 \cup C_2$ and strongly anticomplete to $C_3 \cup C_4 \cup C_5$. Therefore, by the maximality of Y_1, $x \in Y_1$, a contradiction. This proves that $V(T) = C \cup Y \cup Z$.

The following claim proves property (1-d):

(i) *If $i \neq j$, then Y_i is strongly anticomplete to Y_j.*

Let $c_j \in C_j$ with $j \in [5]$. If there exist adjacent $y_i \in Y_i$ and $y_{i+1} \in Y_{i+1}$ for some i, then y_i-y_{i+1}-c_{i+2}-c_{i+3}-c_{i+4}-c_i-y_i, is a weakly induced cycle of length six, a contradiction. If there exist adjacent $y_i \in Y_i$ and $y_{i+2} \in Y_{i+2}$ for some i, then y_i-y_{i+2}-c_{i+2}-c_{i+1}-y_i is a weakly induced cycle and c_{i+4} has no neighbor in it, contrary to (5.2.5). Using the symmetry, this proves **(i)**. □

The following claim proves properties (2-a) and (2-b):

(ii) *Suppose that $Y_i \neq \emptyset$. Then, Y_i is strongly complete to $Z_{i+2} \cup Z_{i+4}$ and strongly anticomplete to $Z_i \cup Z_{i+1} \cup Z_{i+3}$. Moreover, $C_i \cup C_{i+1} \cup Z_{i+2} \cup Z_{i+4}$ is a strong clique.*

Let $y \in Y_i$. Suppose that y is adjacent to $z \in Z_i \cup Z_{i+1} \cup Z_{i+3}$. It follows from the definition of Z_j, $j = i, i+1, i+3$ that z has neighbors $c_{i+2} \in C_{i+2}$ and $c_{i+4} \in C_{i+4}$. But now, z is complete to the triad $\{y, c_{i+2}, c_{i+4}\}$, a contradiction. This proves that Y_i is strongly anticomplete to $Z_i \cup Z_{i+1} \cup Z_{i+3}$. Next, suppose that y is antiadjacent to $z' \in Z_{i+2} \cup Z_{i+4}$. From the symmetry, we may assume that $z' \in Z_{i+2}$. But now, c_{i+1} is complete to the triad $\{c_{i+2}, z', y\}$, a contradiction. This proves that Y_i is strongly complete to $Z_{i+2} \cup Z_{i+4}$.

For $j \in [5]$, let $c_j \in C_j$. C_i is strongly complete to C_{i+1} because if there exist antiadjacent $c'_i \in C_i$ and $c'_{i+1} \in C_{i+1}$, then c'_i-y-c'_{i+1}-c_{i+2}-c_{i+3}-c_{i+4}-c'_i is a weakly induced cycle of length six, a contradiction. It follows from the definition of Z_{i+2} that C_i is strongly complete to Z_{i+2}. C_i is strongly complete to Z_{i+4}, because if there exist antiadjacent $c'_i \in C_i$ and $z_{i+4} \in Z_{i+4}$, then $T|\{c'_i, c_{i+1}, \dots, c_{i+4}, y, z_{i+4}\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph. From the symmetry, it follows that C_{i+1} is strongly complete to $Z_{i+2} \cup Z_{i+4}$. Finally, suppose that there exist antiadjacent $z_{i+2} \in Z_{i+2}$ and $z_{i+4} \in Z_{i+4}$. Let $c'_{i+3} \in C_{i+3}$ be a neighbor of z_{i+2}. If c'_{i+3} is antiadjacent to z_{i+4}, then $T|\{c_i, c_{i+2}, c'_{i+3}, c_{i+4}, z_{i+2}, z_{i+4}, y\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph. Thus, c'_{i+3} is adjacent to z_{i+4}. But now, $T|\{c_i, c_{i+1}, c_{i+2}, c'_{i+3}, c_{i+4}, z_{i+2}, z_{i+4}, y\}$ contains $\mathcal{G}_3$ as a weakly induced subgraph, a contradiction. This proves that Z_{i+2} is strongly complete to Z_{i+4}. Now the **(ii)** follows from the symmetry. □

This proves (5.4.10). ■

(5.4.11) *Let T be an $\mathcal{F}$-free long circular interval trigraph. Assume that T has a semihole of length seven. Then, $V(T)$ can be partitioned into 14 strong cliques $C_1, \dots, C_7$, $Y_1, \dots, Y_7$ such that*

(a) *C_i is complete to C_{i+1} and strongly anticomplete to C_j with $j \notin \{i-1, i, i+1\}$,*

(b) *Y_i is strongly complete to $C_i \cup C_{i+1}$ and strongly anticomplete to C_j with $j \notin \{i, i+1\}$,*

(c) *Y_i is strongly anticomplete to Y_j for $i \neq j$.*

Proof. Let $C_1, \dots, C_7$ be cliques such that C_i is complete to C_{i+1} and strongly anticomplete to C_j with $j \notin \{i-1, i, i+1\}$, and let $C = \bigcup_{i=1}^{7} C_i$ be maximal. For $i = 1, \dots, 7$, let Y_i be the vertices in $V(T) \setminus C$ that are strongly complete to $C_i \cup C_{i+1}$ and strongly anticomplete to C_j with $j \notin \{i, i+1\}$, and let $Y = \bigcup_{i=1}^{7} Y_i$. It follows from the fact that T has a semihole of length seven that $C_i \neq \emptyset$ for

$i \in [7]$. Furthermore, since T is claw-free it follows that each C_i and each Y_i is a strong clique.

We claim that $V(T) = C \cup Y$. For suppose for a contradiction that there exists $x \in V(T) \setminus (C \cup Y)$. In what follows, we say that $F = f_1\text{-}f_2\text{-}\dots\text{-}f_7\text{-}f_1$ is an *aligned semihole in* C if $f_i \in C_i$ for all $i \in [7]$. It follows from (5.4.9) that, for every aligned semihole F in C, x is either a hat or a clone for F. First suppose that x is a hat of type i, say, for some aligned semihole $F = f_1\text{-}f_2\text{-}\dots\text{-}f_7\text{-}f_1$ in C. From the symmetry, we may assume that $i = 1$. We claim that x is strongly anticomplete to C_3. For suppose that x has a neighbor $c_3 \in C_3$. Then, $T|(V(F) \cup \{x, c_3\}$ contains $\mathcal{G}_2$ as a weakly induced subgraph, a contradiction. Therefore, x is strongly anticomplete to C_3, and by symmetry x is strongly anticomplete to C_7. By rerouting F, it follows from (5.4.9) that x is strongly anticomplete to $C_4 \cup C_5 \cup C_6$. Next, again by rerouting F, it follows from (5.4.9) that x is strongly complete to $C_1 \cup C_2$ and, by the maximality of Y_1, $x \in Y_1$, a contradiction. So we may assume that x is not a hat for any aligned semihole in C. Now let $F = f_1\text{-}f_2\text{-}\dots\text{-}f_7\text{-}f_1$ be an aligned semihole in C. It follows that x is a clone of type i, say, for F. We may assume that $i = 1$. By rerouting F, it follows that x is complete to $C_2 \cup C_7$, strongly complete to C_1, and strongly anticomplete to $C_3 \cup C_4 \cup C_5 \cup C_6$. Therefore, by the maximality of C_i, $x \in C_i$, a contradiction. This proves that $V(T) = C \cup Y$.

Now suppose that $y_i \in Y_i$ and $y_j \in Y_j$ $(i \neq j)$ are adjacent. Suppose that $j = i+1$. Let $c_j \in C_j$ for all $j \in [7]$. Then, $T|(V(C) \cup \{y_i, y_j\})$ contains a weakly induced cycle of length eight, a contradiction. Thus, $j \notin \{i+1, i-1\}$. We may assume that $i = 1$ and $2 < j < 5$. Now, $y_i\text{-}c_{i+1}\text{-}c_{i+2}\text{-}\dots\text{-}c_j\text{-}y_j\text{-}y_i$ is a semihole of length at least 4 and c_7 has no neighbor in it, contrary to (5.2.5). This proves that Y_i is strongly anticomplete to Y_j for $i \neq j$, thus completing the proof of (5.4.11). ■

This allows us to deal with long circular interval trigraphs that contain a long semihole:

(5.4.12) *Every $\mathcal{F}$-free long circular interval trigraph that has a semihole of length at least five is resolved.*

Proof. Let T be an $\mathcal{F}$-free long circular interval trigraph. From (5.4.4), we may assume that T is not a linear interval trigraph. By (5.2.12), we may assume that for every semihole in T of length five or more, all adjacent pairs are in fact strongly adjacent.

First suppose that T has a semihole of length seven. Then, let $C_1, \dots, C_7$, $Y_1, \dots, Y_7$ be as in (5.4.11). Since the edges of every semihole in T of length seven are strong edges, it follows that C_i is strongly adjacent to C_{i+1} for all $i \in [7]$. If there exists $y \in Y_i$ for some $i \in [7]$, then it follows that y is a simplicial vertex in T and hence T is resolved by (5.2.9). So we may assume that $Y_i = \emptyset$ for all $i \in [7]$. From (5.2.13), we may assume that T has no clones. It follows that T is a cycle of length seven and, thus, every graphic thickening G of T is imperfect and all maximal stable sets in G have size three. Thus, T is resolved by (5.2.10).

So we may assume that T has a semihole of length five and no semihole of length seven. Then, let $C_1, \dots, C_5$, $Y_1, \dots, Y_5$, $Z_1, \dots, Z_5$ be as in (5.4.10) and let $C = \bigcup_{i=1}^5 C_i$ and $Z = \bigcup_{i=1}^5 Z_i$. Since the edges of every semihole in T of length five are strong edges, it follows that C_i is strongly adjacent to C_{i+1} for all $i \in [5]$. Suppose first that $Y_i \neq \emptyset$ for some i. Let $y_i \in Y_i$. It follows from (5.4.10) that $N[x] = Y_i \cup C_i \cup C_{i+1} \cup Z_{i+2} \cup Z_{i+4}$ and $Y_i \cup C_i \cup C_{i+1} \cup Z_{i+2} \cup Z_{i+4}$ is a strong clique. Hence, y is a simplicial vertex in T and, thus, T is resolved by (5.2.9). So may assume that $Y_i = \emptyset$ for all $i \in [5]$.

If T has no triad, then T is resolved by (5.2.10). Therefore, we may assume that T has a triad $S = \{s_1, s_2, s_3\}$. First suppose that $|S \cap Z| = 3$. From the symmetry, we may assume that $s_1 \in Z_1$, $s_2 \in Z_2$ and $s_3 \in Z_3 \cup Z_4$. It follows from the definition of Z_i that $Z_1 \cup Z_2$ is complete to C_4. Suppose first that $s_3 \in Z_3$. Let $c_4 \in C_4$ be a neighbor of s_3. Now, c_4 is complete to S, a contradiction. It follows that $s_3 \in Z_4$. From the symmetry, we may assume that Z_4 is complete to C_3. Let $c_3 \in C_3$ be a neighbor of s_2. It follows that c_3 is complete to S, a contradiction. Next, suppose that $|S \cap Z| = 2$ and hence $|S \cap C| = 1$. We may assume that $s_1 \in C_1$. It follows from (5.4.10) that C_1 is complete to $Z_3 \cup Z_4$. Hence, from the symmetry, we may assume that $s_2 \in Z_1 \cup Z_2$ and $s_3 \in Z_5$. First suppose that $s_2 \in Z_1$. Let $c_2 \in C_2$ be a neighbor of s_2. Then c_2 is complete to S, a contradiction. It follow that $s_2 \in Z_2$. Let $c_3 \in C_3$ be a neighbor of s_2, and let $c_4 \in C_4$ be a neighbor of s_3. Now, $T|\{s_1, c_2, c_3, c_4, c_5, s_2, s_3\}$, where $c_2 \in C_2$ and $c_5 \in C_5$, contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. Therefore, since $T|C$ contains no triad, it follows that $|S \cap Z| = 1$ and $|S \cap C| = 2$. From the symmetry, we may assume that $s_1 \in C_1$ and $s_2 \in C_3$. Because C_1 is strongly complete to $Z_3 \cup Z_4$, and C_3 is strongly complete to $Z_1 \cup Z_5$, it follows that $s_3 \in Z_2$. But this contradicts the fact that Z_2 is strongly complete to at least one of C_1, C_3. This proves (5.4.12). ■

The previous two lemmas imply the main result of this section:

(5.4.13) *Every $\mathcal{F}$-free long circular interval trigraph is resolved.*

Proof. Let T be a $\mathcal{F}$-free long circular interval trigraph. If T is a linear interval trigraph, then it follows from (5.4.4) that T is resolved. If T has a semihole of length at least five, then T is resolved by (5.4.12). Therefore, we may assume that T has no semihole of length at least five and, thus, the result follows from (5.4.8). This proves (5.4.13). ■

5.4.3 $\mathcal{F}$-free three-cliqued trigraphs

In this section, we deal with three-cliqued claw-free trigraphs. The approach is as follows. Theorem 5.2.4 states that every three-cliqued claw-free trigraph either lies in $\mathcal{TC}_1 \cup \mathcal{TC}_2 \cup \ldots \cup \mathcal{TC}_5$, or admits a worn hex-chain of trigraphs in $\mathcal{TC}_1 \cup \mathcal{TC}_2 \cup \ldots \cup \mathcal{TC}_5$. We first show that in the context of $\mathcal{F}$-free three-cliqued claw-free trigraphs, it suffices to consider only the basic three-cliqued claw-free trigraphs, and basic three-cliqued claw-free trigraphs that are hex-joined with a strong clique. After having stated and proved this result, we will go through the remaining cases and conclude that $\mathcal{F}$-free three-cliqued claw-free trigraphs are resolved.

A three-cliqued claw-free trigraph (T, A, B, C) is called *very basic* if $(T, A, B, C) \in \mathcal{TC}_1 \cup \mathcal{TC}_2 \cup \mathcal{TC}_3 \cup \mathcal{TC}_5$. We start with the following lemma, which states that it suffices to consider three-cliqued claw-free trigraphs that are very basic, or that are a hex-join of a very basic three-cliqued claw-free trigraph and a strong clique.

(5.4.14) *Let (T, A, B, C) be an $\mathcal{F}$-free three-cliqued claw-free trigraph. Then, either T is resolved* *or (T, A, B, C) is*

(a) *a very basic three-cliqued claw-free trigraph, or*

(b) *a trigraph that is the hex-join of a very basic three-cliqued claw-free trigraph and a strong clique.*

Proof. We may assume that (T, A, B, C) is not very basic. Thus, (T, A, B, C) admits a worn hex-chain. We may assume that T is not resolved. We start with two claims about worn hex-joins.

(i) *Suppose that (T, A, B, C) is a worn hex-join of two three-cliqued claw-free trigraphs (T_1, A_1, B_1, C_1) and (T_2, A_2, B_2, C_2). Then, at least one of T_1, T_2 does not contain a triad.*

Suppose that for $i = 1, 2$, T_i contains a triad $\{a_i, b_i, c_i\}$. From the symmetry and the fact that A_i, B_i, C_i are strong cliques, we may assume that for $i = 1, 2$, $a_i \in A_i$, $b_i \in B_i$ and $c_i \in C_i$. But now a_1-a_2-b_1-b_2-c_1-c_2-a_1 is a weakly induced cycle of length six in T, a contradiction. This proves **(i)**. □

(ii) *A worn hex-chain of antiprismatic three-cliqued claw-free trigraphs is antiprismatic.*

Since a worn hex-chain can be constructed by iteratively hex-joining two trigraphs, it suffices to show the lemma for worn hex-joins. So, for $i = 1, 2$, let (T_i, A_i, B_i, C_i), be an antiprismatic three-cliqued claw-free trigraph and consider the worn hex-join T' of (T_1, A_1, B_1, C_1) and (T_2, A_2, B_2, C_2). In order to show that T' is antiprismatic, it suffices to show that for every triad S in T', every vertex $v \in V(T') \setminus S$ has at least two strong neighbors in S. So let S be a triad in T'. From the symmetry, we may assume that S has at least one vertex in T_1. From the definition of a worn hex-join, and the fact that A_1, B_1, C_1 are strong cliques, it follows that $S = \{a, b, c\}$ with $a \in A_1, b \in B_1, c \in C_1$. Now let $v \in V(T') \setminus S$. If $v \in V(T_1)$, then it follows from the fact that T_1 is antiprismatic that v has at least two strong neighbors in S. So we may assume that $v \in V(T_2)$, and from the symmetry we may assume that $v \in A_2$. Now v is strongly complete to $A_1 \cup B_1$, and hence v is strongly adjacent to a and b, and antiadjacent to c. This proves **(ii)**. □

First, notice that every very basic three-cliqued claw-free trigraph contains a triad. Hence, it follows from **(i)**, Theorem 5.2.4 and the symmetry that we may assume that (T, A, B, C) admits a worn hex-chain into terms, at most one of which is a basic three-cliqued claw-free trigraph, and whose other terms are three-cliqued claw-free trigraphs with no triad (and, in particular, they are antiprismatic). If all terms are antiprismatic three-cliqued claw-free trigraphs, then T is antiprismatic by **(ii)** and thus the lemma holds by (5.4.3). So we may assume that exactly one of the terms is a very basic three-cliqued claw-free trigraph. Notice that a worn hex-chain of antiprismatic three-cliqued claw-free trigraphs is an antiprismatic three-cliqued claw-free trigraph. Possibly by taking together all terms that are antiprismatic three-cliqued claw-free trigraphs, it follows that T is a worn hex-join of a very basic three-cliqued claw-free trigraph L, and an antiprismatic three-cliqued claw-free trigraph R that contains no triad. Since every vertex of a very basic three-cliqued claw-free trigraph is in a triad, it follows that T is not only a worn hex-join, but in fact a hex-join of a very basic three-cliqued claw-free trigraph $L = (L_1, L_2, L_3)$, and an antiprismatic three-cliqued claw-free trigraph $R = (R_1, R_2, R_3)$ that contains no triad. We may assume that for $\{i, j, k\} = \{1, 2, 3\}$, R_i is strongly anticomplete to L_i and strongly complete to $L_j \cup L_k$.

(iii) *For $i = 1, 2, 3$, L_i is not strongly anticomplete to $L \setminus L_i$.*

It suffices to show this for $i = 2$. Suppose that L_2 is strongly anticomplete to $L \setminus L_2$. First suppose that L_1 is strongly anticomplete to L_3. Then L is a disjoint union of strong cliques and, by (5.2.13) applied to L, we may assume that L is a triad, and thus that L is antiprismatic, a contradiction. Hence, L_1 is not strongly anticomplete to L_3. Let $l_2 \in L_2$. Since l_2 is not simplicial, there exist antiadjacent $r_1 \in R_1$ and $r_3 \in R_3$. Now (L_1, L_3) is a homogeneous pair of cliques in T such that L_1 is neither strongly complete nor strongly anticomplete to L_3, and r_1-l_2-r_3 is a weakly induced path that contradicts (5.2.11). This proves **(iii)**. □

(iv) *Suppose that there exist antiadjacent $r_1 \in R_1$ and $r_2 \in R_2$. Then,*

(iv-a) *there is no weakly induced path x_1-x_2-x_3-x_4-x_5 with $x_1 \in L_2$, $x_2, x_3 \in L_1$ and $x_4, x_5 \in L_3$, or with $x_1 \in L_1$, $x_2, x_3 \in L_2$ and $x_4, x_5 \in L_3$;*

(iv-b) *there is no triad $\{l_1, l_2, l_3\}$ with $l_i \in L_i$ such that l_1 and l_2 are semiadjacent;*

(iv-c) *if $l_1 \in L_1$ is adjacent to $l_3 \in L_3$, and $l_2 \in L_2$ is in a triad with l_1, then l_2 is strongly anticomplete to L_1.*

For part **(iv-a)**, suppose that there exist such $r_1, r_2, x_1, \dots, x_5$. Then, $T|\{x_1, r_1, x_4, r_2, x_2, x_3, x_5\}$ contains $\mathcal{G}_1$ as weakly induced subgraph, a contradiction. This proves **(iv-a)**.

For part **(iv-b)**, suppose that such l_1, l_2, l_3 exist. Then, l_1-l_2-r_1-l_3-r_2-l_1 is a weakly induced cycle of length five that contradicts (5.2.12). This proves **(iv-b)**.

For part **(iv-c)**, let $l_1 \in L_1$ and $l_3 \in L_3$ be adjacent, and let $l_2 \in L_2$ be in a triad with l_1. Suppose first that $\{l_1, l_2, l_3\}$ is a triad. It follows that l_1 is semiadjacent to l_3 and l_2 is strongly antiadjacent to l_1 and l_3. We may assume that l_2 has a neighbor $l_1' \in L_1$. Because l_1' is not complete to the triad $\{l_1, l_2, l_3\}$, it follows that l_1' is strongly antiadjacent to l_3. But now, l_1-l_3-r_1-l_2-l_1'-l_1 is a weakly induced cycle in T that contradicts (5.2.12). This proves that $\{l_1, l_2, l_3\}$ is not a triad.

Let $\{l_1, l_2, l_3'\}$ be a triad. It follows that $l_3' \neq l_3$. It follows from **(iv-b)** that $l_1 l_2$ is not a semiedge and thus l_1 is strongly antiadjacent to l_2. Since l_3 is not complete to $\{l_1, l_2, l_3'\}$, it follows that l_3 is strongly antiadjacent to l_2. Because $\{l_1, l_2, l_3\}$ is not a triad, it follows that l_1 is strongly adjacent to l_3. Let $l_1' \in L_1$ be a nonneighbor of l_3 (l_1' exists because l_3 is in a triad). Suppose first that l_1' is adjacent to l_2. Because l_1' is not complete to $\{l_1, l_2, l_3'\}$, l_1' is strongly antiadjacent to l_3'. But now l_2-l_1'-l_1-l_3-l_3' is a weakly induced path contradicting **(iv-a)**. This proves that l_1' is strongly antiadjacent to l_2. We may assume that l_2 has a neighbor $l_1'' \in L_1$. Because l_1'' is not complete to $\{l_1', l_2, l_3\}$ and not complete to $\{l_1, l_2, l_3'\}$, it follows that l_1'' is strongly anticomplete to $\{l_3, l_3'\}$. But now l_2-l_1''-l_1-l_3-l_3' is a weakly induced path that contradicts **(iv-a)**. This proves **(iv-c)**, thus completing the proof of **(iv)**. □

(v) *At least one of the pairs (R_1, R_2), (R_2, R_3), (R_1, R_3) is strongly complete.*

We first claim that every vertex of R_2 is strongly complete to at least one of R_1, R_3. For suppose that there exists $r_2 \in R_2$ with antineighbors $r_1 \in R_1$ and $r_3 \in R_3$. Since R contains no triad, it follows that r_1 is strongly adjacent to r_3. It follows from **(iii)** that L_2 is not strongly anticomplete to $L_1 \cup L_3$ and thus, from the symmetry, we may assume that there exist adjacent $l_1 \in L_1$ and $l_2 \in L_2$. Let $\{l'_1, l_2, l_3\}$ be a triad containing l_2. If $l'_1 = l_1$, then it follows that l_1 is semiadjacent to l_2, thus contradicting **(iv-b)**. Thus, $l_1 \neq l'_1$. Since l_1 is not complete to the triad $\{l'_1, l_2, l_3\}$, it follows that l_1 is strongly antiadjacent to l_3. But now $T|\{l_3, r_2, l_1, r_3, r_1, l'_1, l_2\}$ contains $\mathcal{G}_1$ as weakly induced subgraph. This proves the claim. Notice that by symmetry it follows that for $\{i, j, k\} = \{1, 2, 3\}$, every vertex of R_i is strongly complete to at least one of R_j, R_k.

Suppose that there exist antiadjacent pairs (r_1, r'_2), (r_2, r'_3), (r'_1, r_3) with $r_i, r'_i \in R_i$. It follows from our previous claim that $r_i \neq r'_i$ for $i = 1, 2, 3$, and all pairs except (r_1, r'_2), (r_2, r'_3), (r'_1, r_3) are strongly adjacent. Let $\{l_1, l_2, l_3\}$ with $l_i \in L_i$ be a triad. Now, $T|\{l_1, l_2, l_3, r_1, r'_1, r_2, r'_2, r_3, r'_3\}$ contains $\mathcal{G}_4$ as a weakly induced subgraph, a contradiction. This proves **(v)**. □

By **(v)**, we may assume that R_1 is strongly complete to R_3. We may assume that R is not a strong clique and thus we may assume that there exist antiadjacent $r_1 \in R_1$ and $r_2 \in R_2$.

(vi) *No vertex in L_1 has both a neighbor in L_2 and a neighbor in L_3.*

Suppose that $l_1 \in L_1$ has a neighbor $l_3 \in L_3$. Let $l_2 \in L_2$ be in a triad with l_1. By **(iv-c)**, l_2 is strongly anticomplete to L_1. Since l_2 is not simplicial, l_2 has a neighbor in L_3. Now, from the symmetry between L_1 and L_2 and by **(iv-c)**, it follows that l_1 is strongly anticomplete to L_2. This proves **(vi)**. □

We may assume that $K = R_1 \cup L_2 \cup R_3$ is not a dominant clique in T. Thus, there exists a stable set $S \subseteq (V(T) \setminus K)$ that covers K. First suppose that $S \cap R_2 \neq \emptyset$. Then, since R_2 is strongly complete to $L_1 \cup L_3$, it follows that $S \subseteq R_2$. But now, S does not cover L_2, a contradiction. Therefore, $S \cap R_2 = \emptyset$. It follows that $S \subseteq L_1 \cup L_3$. Suppose next that $S \subseteq L_1$. Let l_1 be the unique vertex in S, and let $\{l_1, l_2, l_3\}$ be a triad. Clearly, $\{l_1, l_2, l_3\}$ is a larger stable set than S, a contradiction. From this and from the symmetry, it follows that $S = \{l_1, l_3\}$ with $l_1 \in L_1$ and $l_3 \in L_3$.

Let $z \in L_2$. By the maximality of S, it follows that l_1 and l_3 are not both antiadjacent to z. This proves that for every $z \in L_2$, z is strongly adjacent to at least one of l_1, l_3.

Let $l_2, l'_2 \in L_2$ be antineighbors of l_1, l_3, respectively. Notice that l_2, l'_2 exist since each vertex in L is in a triad. It follows by the previous argument that $l_2 \neq l'_2$, l_1 is strongly adjacent to l'_2, and l_3 is strongly adjacent to l_2. Let $l'_3 \in L_3$ be an antineighbor of l_2. It follows from **(vi)** that l'_3 is strongly antiadjacent to l_1. Because l'_2 is not complete the triad $\{l_1, l_2, l'_3\}$, it follows that l'_2 is strongly antiadjacent to l'_3. But now, l_1-l'_2-l_2-l_3-l'_3 is a weakly induced path that contradicts **(iv-a)**. Thus K is a dominant clique, a contradiction. This proves that R is a strong clique, and hence this proves (5.4.14). ■

Recall that the $\mathcal{F}$-free three-cliqued claw-free trigraphs that remain open after (5.4.14) are the very basic three-cliqued claw-free trigraphs, and the hex-joins of very basic three-cliqued claw-free trigraphs with a strong clique. The next few lemmas deal with these cases. We start with three-cliqued claw-free trigraphs where the part that is very basic is a type of line trigraph.

(5.4.15) *No three-cliqued claw-free trigraph in $\mathcal{TC}_1$ is $\mathcal{F}$-free.*

Proof. Let $(T, L_1, L_2, L_3) \in \mathcal{TC}_1$. Let H, v_1, v_2, v_3 be as in the definition of $\mathcal{TC}_1$ with respect to T. First observe that if H contains a cycle of length six (not necessarily induced), then, by the definition of a line trigraph, T contains a weakly induced cycle of length six, and thus the lemma holds. So we may assume now that H does not contain any cycle of length six.

For $i = 1, 2, 3$, let W_i be the vertices of $V(H) \setminus \{v_1, v_2, v_3\}$ that are complete to $\{v_1, v_2, v_3\} \setminus \{v_i\}$ and nonadjacent to v_i, and let Z be the vertices that are complete to $\{v_1, v_2, v_3\}$. It follows from the definition of $\mathcal{TC}_1$ that $|W_i| \leq 1$ for all i. Also, if $|Z| \geq 3$, say $z_1, z_2, z_3 \in Z$, then $H|\{z_1, z_2, z_3, v_1, v_2, v_3\}$ contains a cycle of length six, a contradiction. Thus, we may assume that $|Z| \leq 2$.

If W_1, W_2, W_3 are all nonempty, say $w_i \in W_i$ for $i = 1, 2, 3$, then $H|\{v_1, v_2, v_3, w_1, w_2, w_3\}$ contains a cycle of length six, a contradiction. By symmetry, we may assume that $W_2 = \emptyset$. Now, from the fact that $|W_3| \leq 1$, $|Z| \leq 2$, and $\deg_H(v_1) \geq 3$, it follows that $|W_3| = 1$ and $|Z| = 2$. From the symmetry, it follows that $|W_1| = 1$. Let $W_i = \{w_i\}$ for $i = 1, 3$ and $Z = \{z_1, z_2\}$. But now, $H|\{v_1, v_2, v_3, w_1, w_3, z_1\}$ contains a cycle of length six, a contradiction. This proves (5.4.15). ■

Next, we deal with three-cliqued claw-free trigraphs where the part that is very basic is a long circular interval trigraph. We first prove the following lemma.

(5.4.16) *Every $(T, L_1, L_2, L_3) \in \mathcal{TC}_2$ contains a semihole of length at least five.*

Proof. Suppose that T has no induced semihole of length at least five. It follows from (5.4.7) and the definition of $\mathcal{TC}_2$ that T is either of the $\bar{C}_7$ type, or T admits a C_4-structure. If T is of the $\bar{C}_7$ type, then it follows from (5.4.6) that T has no triad, a contradiction. So we may assume that T admits a C_4-structure $(A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4)$. Recall that every vertex in T is in a triad and that T contains no four pairwise antiadjacent vertices.

(i) *For $i \in [4]$, if $a_i \in A_i$ is strongly complete to A_{i+1}, then $B_{i+1} \neq \emptyset$.*

Let $i \in [4]$, let $a_i \in A_i$ be strongly complete to A_{i+1}, and suppose that $B_{i+1} = \emptyset$. Let $S = \{a_i, s_1, s_2\}$ be a triad in T. Since a_i is strongly complete to $A_{i+1} \cup B_i \cup B_{i+4}$, and $B_{i+1} = \emptyset$, it follows that $\{s_1, s_2\} \subseteq A_{i+2} \cup A_{i+3} \cup B_{i+2}$. First suppose that $s_1 \in A_{i+2}$. Because S is a triad and A_{i+2} is strongly complete to B_{i+2}, it follows that $s_2 \in A_{i+3}$. But now, $s_2 \in A_{i+3}$ has a nonneighbor in both A_i and A_{i+2}, a contradiction. Thus, we may assume that $S \cap A_{i+2} = \emptyset$. It follows that we may assume that $s_1 \in A_{i+3}$ and $s_2 \in B_{i+2}$. But this contradicts the fact that A_{i+3} is strongly complete to B_{i+2}. This proves **(i)**. □

First suppose that for $i \in [4]$, A_i is strongly complete to A_{i+1}. Then, it follows from **(i)** that $B_i \neq \emptyset$ for all $i \in [4]$. But now, $\{b_1, b_2, b_3, b_4\}$ is a set of four pairwise antiadjacent vertices, a contradiction.

Thus, we may assume that, for some $i \in [4]$, there exist antiadjacent $a_i \in A_i$ and $a_i \in A_{i+1}$. It follows from the definition of a C_4-structure that a_i is strongly complete to A_{i+3} and a_{i+1} is strongly complete to A_{i+2}. Thus, it follows from **(i)** applied to a_i and A_{i+3} that there exists $b_{i+2} \in B_{i+2}$. If there exist semiadjacent $a_{i+2} \in A_{i+2}$ and $a_{i+3} \in A_{i+3}$, then it follows from the symmetry that there exists $b_i \in B_i$, but now a_i-b_i-a_{i+1}-a_{i+2}-b_{i+2}-a_{i+3}-a_i is a weakly induced cycle of length six, a contradiction. Therefore, A_{i+2} is strongly complete to A_{i+3}. Thus, it follows from **(i)** applied to A_{i+2} and A_{i+3} that there exists $b_{i+3} \in B_{i+3}$ and, symmetrically, there exists $b_{i+1} \in B_{i+1}$. Since T has no weakly induced cycle of length six, it follows that at least one of the pairs (A_{i+1}, A_{i+2}) and (A_i, A_{i+3}) is strongly complete. We may assume that A_{i+1} is strongly complete to A_{i+2}. Now, it follows from **(i)** that there exists $b_i \in B_i$. But now, $\{b_1, b_2, b_3, b_4\}$ is a set of four pairwise antiadjacent vertices, a contradiction. This proves (5.4.16). ■

This enables us to deal with three-cliqued claw-free trigraphs where the part that is very basic is a long circular interval trigraph.

(5.4.17) *Let T be an $\mathcal{F}$-free trigraph that is a hex-join of $(T_1, L_1, L_2, L_3) \in \mathcal{TC}_2$ and (T_2, R_1, R_2, R_3), where $R_1 \cup R_2 \cup R_3$ is a strong clique. Then T is resolved.*

Proof. It follows from (5.2.13) that we may assume that $|R_i| \leq 1$ for $i = 1, 2, 3$. Next, we note that if $|R_1 \cup R_2 \cup R_3| < 3$, then T is a long circular interval trigraph, and the lemma holds by (5.4.13). So we may assume that $|R_i| = 1$ for $i = 1, 2, 3$. Let $R_i = \{r_i\}$, for $i = 1, 2, 3$. It follows from (5.4.16) that T_1 has a semihole of length at least five. It follows from the fact that T_1 is a three-cliqued claw-free trigraph that T_1 has no semihole of length seven. Thus, since T_1 is $\mathcal{F}$-free, it follows that T_1 contains a semihole of length five. Let $C_1, \dots, C_5$, $Y_1, \dots, Y_5$, $Z_1, \dots, Z_5$ be as in (5.4.10). If there are semiadjacent $c_i \in C_i$ and $c_{i+1} \in C_{i+1}$, then it follows from (5.2.12) that T is resolved. So we may assume that C_i is strongly complete to C_{i+1} for all $i \in [5]$. If $Y_i = \emptyset$ for all i, then it follows from the proof of (5.4.12) that T has no triad, a contradiction. So from the symmetry we may assume that $Y_1 \neq \emptyset$. Recall that (T_1, L_1, L_2, L_3) is a three-cliqued claw-free trigraph. The following claim shows how $C_1, \dots, C_5$, and Y_1 relate to the three cliques L_1, L_2, L_3.

(i) *Up to symmetry, $Y_1 \cup C_1 \cup C_2 \subseteq L_1$, $C_3 \subseteq L_2$, $C_4 \subseteq L_2 \cup L_3$, and $C_5 \subseteq L_3$.*

Let $y_1 \in Y_1$. We may assume that $y_1 \in L_1$. Since L_1, L_2 and L_3 are strong cliques, it follows from the symmetry that we may assume that $C_3 \subseteq L_2$, $C_5 \subseteq L_3$, and $C_4 \subseteq L_2 \cup L_3$. Therefore, it follows that $Y_1 \subseteq L_1$. Now, let $c_4 \in C_4$. From the symmetry, we may assume that $c_4 \in L_2$. It follows that $C_2 \subseteq L_1$. We claim that $C_1 \subseteq L_1$. For suppose not. Then, since L_2 is a strong clique, it follows that there exists $c_1 \in C_1$ such that $c_1 \in L_3$. For $i = 2, 3, 5$, let $c_i \in C_i$. Now, $T|\{c_1, c_2, c_3, c_4, c_5, y_1, r_3\}$ is weakly isomorphic to $\mathcal{G}_1$, a contradiction. Thus, $C_1 \subseteq L_1$ and **(i)** holds. □

It follows from **(i)** that we may assume that $Y_1 \cup C_1 \cup C_2 \subseteq L_1$, $C_3 \subseteq L_2$, $C_4 \subseteq L_2 \cup L_3$, and $C_5 \subseteq L_3$ Let $y_1 \in Y_1$. We claim that y_1 is a simplicial vertex in T. It follows from (5.4.10) that $N[y_1] = Y_1 \cup C_1 \cup C_2 \cup Z_3 \cup Z_5 \cup \{r_2, r_3\}$ and $N[Y_1] \setminus \{r_2, r_3\}$ is a strong clique. From this, and from the symmetry, it suffices to show that $Y_1 \cup Z_3$ is strongly complete to $\{r_2, r_3\}$. Since $C_1 \cup C_2 \subseteq L_1$, it follows immediately from the definition of a hex-join that $C_1 \cup C_2$ is strongly complete to $\{r_2, r_3\}$. So let $z_3 \in Z_3$. Let $c_j \in C_j$ for $j \in [5]$. If z_3 is antiadjacent to r_2, then c_2 is complete to the triad $\{c_3, r_2, z_3\}$, a contradiction. Thus, z_3 is strongly adjacent to r_2. Now suppose that z_3 is antiadjacent to r_3. If r_3 is adjacent to c_4, then $T|\{c_1, c_2, \dots, c_5, z_3, r_3, y_1\}$ contains $\mathcal{G}_3$ as weakly induced subgraph, a contradiction. If r_3 is antiadjacent to c_4, then $T|\{c_1, c_3, c_4, c_5, z_3, r_3, y_1\}$ contains $\mathcal{G}_1$ as weakly induced subgraph, a contradiction. This proves that Z_3 is strongly complete to $\{r_2, r_3\}$ and, from the symmetry, that Z_5 is strongly complete to $\{r_2, r_3\}$ Thus, $N[y_1]$ is a strong clique, hence y_1 is a simplicial vertex in T and the lemma holds by (5.2.9). This proves (5.4.17). ■

The next lemma deals with three-cliqued claw-free trigraphs where the part that is very basic is a near-antiprismatic trigraph.

(5.4.18) *Let T be an $\mathcal{F}$-free trigraph that is a hex-join of $(T_1, L_1, L_2, L_3) \in \mathcal{TC}_3$ and (T_2, R_1, R_2, R_3), where $R_1 \cup R_2 \cup R_3$ is strong clique. Then T is resolved.*

Proof. Let $(T_1, L_1, L_2, L_3) \in \mathcal{TC}_3$ and let a_0, b_0, A, B, C, X, n be as in the definition of a near-antiprismatic trigraph. Notice that $L_1 = A \setminus X$, $L_2 = B \setminus X$, $L_3 = C \setminus X$. If a_0 is strongly antiadjacent to b_0, then $N(a_0) = L_1 \cup (R_2 \cup R_3)$, hence a_0 is a simplicial vertex and the lemma holds by (5.2.9). So we may assume that a_0 is semiadjacent to b_0. First suppose that there exist antiadjacent $a_i \in L_1$ and $b_j \in L_2$, for $i, j \leq n$ and $i \neq j$. Because $|L_3| \geq 2$, it follows that both a_i and b_j have a neighbor in L_3. Therefore, there exists an shortest weakly induced path P from a_i to b_j with interior in L_3. Now, (5.2.12) applied to a_0-a_i-P^*-b_j-b_0-a_0 implies that T is resolved.

Thus, we may assume that L_1 is strongly complete to L_2. It follows from the definition of $\mathcal{TC}_3$ that $L_1 = \{a_1\}$, $L_2 = \{b_1\}$, and hence that $n = 2$ and $L_3 = \{c_1, c_2\}$. Moreover, c_1 is strongly anticomplete to $\{a_1, b_1\}$. Therefore, $N(c_1) = \{c_2\} \cup R_1 \cup R_2$, which is a strong clique. Thus, c_1 is a simplicial vertex and T is resolved by (5.2.9). This proves (5.4.18). ■

Finally, we deal with trigraphs where the part that is very basic is a sporadic exception.

(5.4.19) *Let T be an $\mathcal{F}$-free trigraph T that is a hex-join of $(T_1, L_1, L_2, L_3) \in \mathcal{TC}_5$ and (T_2, R_1, R_2, R_3), where $R_1 \cup R_2 \cup R_3$ is strong clique. Then T is resolved.*

Proof. First suppose that T_1 is of the first type of sporadic trigraphs. Let v_1, ..., v_8, A, B, C, X be as in the definition of T_1. Observe that $L_1 = A \setminus X$, $L_2 = B \setminus X$, $L_3 = C$. It follows from the definition of T_1 and a hex-join that $N(v_8) = \{v_7\} \cup R_1 \cup R_2$ is a strong clique. Therefore, v_8 is a simplicial vertex in T and hence T is resolved by (5.2.9).

So we may assume that T_1 is of the second type of sporadic trigraphs. Let $v_1, \dots, v_9$ be as in the definition of T_1. Let $j \in \{3, 4\}$ be largest such that v_2 is adjacent to v_j and let $k \in \{5, 6\}$ be smallest such that v_7 is adjacent to v_k. Such j, k exist by the fact that v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$ and v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$. But now v_1-v_2-v_j-v_k-v_7-v_8-v_1 is a weakly induced cycle of length six in T, a contradiction. This proves (5.4.19). ■

This allows us to prove that $\mathcal{F}$-free three-cliqued claw-free trigraphs are resolved:

(5.4.20) *Every $\mathcal{F}$-free three-cliqued claw-free trigraph is resolved.*

Proof. Let (T, A, B, C) be a three-cliqued claw-free trigraph. It follows from (5.4.14) that either T is resolved and the lemma holds, or (T, A, B, C) is very basic, or (T, A, B, C) is a hex-join of a very basic trigraph and a strong clique. We may assume that the former outcome does not hold. If (T, A, B, C) is very basic, we set $(T', A', B', C') = (T, A, B, C)$. Otherwise, let (T', A', B', C') be such that T is a hex-join of a very basic trigraph (T', A', B', C') and a strong clique. Since $(T', A', B', C') \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_5$, the lemma follows from (5.4.15), (5.4.17), (5.4.18) and (5.4.19). This proves (5.4.20). ■

5.4.4 Proof of (5.4.1)

(5.4.1). *Every $\mathcal{F}$-free basic claw-free trigraph is resolved.*

Proof. Let T be an $\mathcal{F}$-free basic claw-free trigraph. It follows that T is either a trigraph from the icosahedron, or an antiprismatic trigraph, or a long circular interval trigraph, or a trigraph that is the union of three strong cliques. It follows from (5.4.2) that T is not a trigraph from the icosahedron. If T is an antiprismatic trigraph, a long circular interval trigraph, or a trigraph that is the union of three strong cliques, then it follows from (5.4.3), (5.4.13), (5.4.20), respectively, that T is resolved. This proves (5.4.1). ■

5.5 A structure theorem for the multigraph underlying the strip-structure of $\mathcal{F}$-free nonbasic claw-free trigraphs

Let G be a nonbasic claw-free graph. We say that (T, H, η) is a *representation* of G if G is a graphic thickening of T, and (H, η) is a nontrivial strip-structure for T. We say that a representation is *optimal* for G if T is not a thickening of any other claw-free trigraph and, subject to that, H has a maximum number of edges.

Let H be a multigraph. We say that a vertex $x \in V(H)$ is a *cut-vertex of H* if $H \setminus \{x\}$ is disconnected. A multigraph H is *2-connected* if H has no cut-vertex. A maximal submultigraph of H that has no cut-vertex is called a *block* of H, and the collection $(B_1, \dots, B_q)$ of blocks of H is called the *block-decomposition of H*. It is well-known that the block-decomposition of a multigraph exists and is unique (see *e.g.*, **[58]**). Observe that a multigraph H is 2-connected if and only if H has at most one block. For a cycle C in H and $F \in E(C)$, let $C \setminus F$ denote the graph obtained from C by deleting F.

Let G be a graph and let $x \in V(G)$. Construct G' by adding a vertex x' such that $N(x) = N(x')$. Then, we say x and x' are *nonadjacent* clones in G' and we say that G' is constructed from G by *nonadjacent cloning of x*. Let $t \geq 1$. Let K_t be a complete graph on t vertices. Let $K_{2,t}$ denote a complete bipartite graph whose vertex set is the union of disjoint stable sets X, Y with $|X| = 2$ and $|Y| = t$. Let $K_{2,t}^{+}$ denote the graph constructed from $K_{2,t}$ by adding an edge between the two vertices in X, where X is as in the definition of $K_{2,t}$.

We define the following two classes of graphs:

$\mathcal{B}_1$: Let us first define the class $\mathcal{B}_1^*$. Let $k \in \{5, 7\}$ and let G be a graph with vertex set $\{c_1, c_2, \dots, c_k\}$ such that c_1-c_2-$\dots c_k$-c_1 is a cycle. If $k = 5$, then each other pair not specified so far is either adjacent or nonadjacent. If $k = 7$, then all pairs are nonadjacent except possibly a subset of the pairs $\{c_1, c_4\}, \{c_1, c_5\}, \{c_4, c_7\}$. Then, $G \in \mathcal{B}_1^*$.

Now let every graph in $\mathcal{B}_1^*$ be in $\mathcal{B}_1$. For every $G' \in \mathcal{B}_1$, let the graph G'' constructed from G' by nonadjacent cloning of a vertex of degree 2 be in $\mathcal{B}_1$.

$\mathcal{B}_2$: Let $\mathcal{B}_2 = \{K_2, K_3, K_4, K_{2,t}, K_{2,t}^{+} \mid t \geq 2\}$.

For a multigraph H, let $U(H)$ be the graph constructed from H by removing all but one in each class of parallel edges and regarding the resulting multigraph as a graph. For $i \in [2]$, we say that a multigraph H is *of the $\mathcal{B}_i$ type* if $U(H)$ is isomorphic to a graph in $\mathcal{B}_i$. It turns out that if (T, H, η) is an optimal representation of an $\mathcal{F}$-free nonbasic claw-free graph, then the structure of H is relatively simple. In particular, the goal of this section is to prove the following:

(5.5.1) *Let G be a graphic thickening of a connected $\mathcal{F}$-free claw-free trigraph that admits a nontrivial strip-structure. Then, G has an optimal representation and, for every optimal representation (T, H, η), all of the following hold:*

(i) *every block of H is either of the $\mathcal{B}_1$ type or of the $\mathcal{B}_2$ type;*

(ii) *at most one block of H is of the $\mathcal{B}_1$ type;*

(iii) *for every cycle C in H with $|E(C)| \geq 4$, all strips of (H, η) at $F \in E(C)$ are spots.*

Figure 5.3 illustrates the structure of H. The block decomposition of the multigraph H shown in the figure has one block of the $\mathcal{B}_1$ type. The other blocks are of the $\mathcal{B}_2$ type.

5.5.1 Properties of optimal representations of $\mathcal{F}$-free nonbasic claw-free graphs

Before we can prove (5.5.1), we need to prove some lemmas. We use the results in this subsection later on as well.

(5.5.2) *Let G be an $\mathcal{F}$-free claw-free graph and let (T, H, η) be an optimal representation of G. Then, for each strip (J, Z), either*

(a) *(J, Z) is a spot, or*

(b) *(J, Z) is isomorphic to a member of $\mathcal{Z}_0$.*

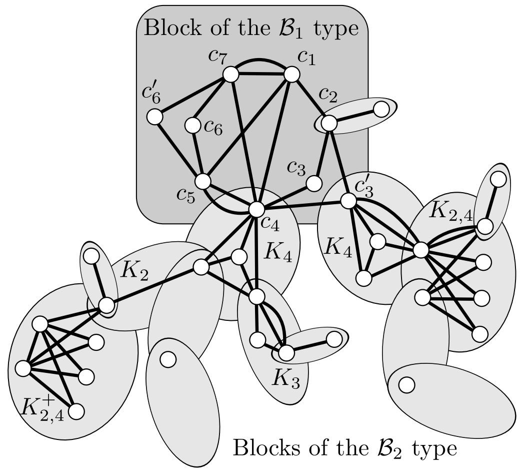


Figure 5.3: *An example of the multigraph H of an optimal representation (T, H, η) of an $\mathcal{F}$-free claw-free trigraph. The ellipses show the blocks of the multigraph. The pendant edges represent strips (J, Z) that satisfy $1 \leq |Z| \leq 2$. All other edges represent strips (J, Z) with $|Z| = 2$.*

Proof. Suppose that, for some $F \in E(H)$, (J, Z) is not a spot and (J, Z) is not isomorphic to a member of $\mathcal{Z}_0$. Then, (J, Z) is a thickening of some member (J', Z') of $\mathcal{Z}_0$. Now, construct (T', H, η') by replacing (J, Z) by (J', Z'), and updating the corresponding sets for η. Then, G is a graphic thickening of T' and T is a thickening of T', contrary to the fact that (T, H, η) is an optimal representation for G. This proves (5.5.2). ■

The following lemma states that T and every strip of the strip-structure is $\mathcal{F}$-free (recall that a trigraph T is $\mathcal{F}$-free if it does not contain any graph in $\mathcal{F}$ as a weakly induced subgraph).

(5.5.3) *Let (T, H, η) be a representation of some $\mathcal{F}$-free claw-free graph G. Then T is $\mathcal{F}$-free and, for all $F \in E(H)$, the strip of (H, η) at F is $\mathcal{F}$-free.*

Proof. It follows from (5.2.1) that if T contains a graph $H \in \mathcal{F}$ as a weakly induced subgraph, then G contains H as an induced subgraph, a contradiction. Therefore, T is $\mathcal{F}$-free. Next, let $F \in E(H)$ and consider the strip (J, Z) of (H, η) at F and suppose that for some $X \subseteq V(J)$, $J|X$ contains a graph $H \in \mathcal{F}$ as a weakly induced subgraph. We may choose X minimal with this property. Because none of the graphs in $\mathcal{F}$ has a simplicial vertex, it follows that $X \cap Z = \emptyset$. Therefore, $J|X$ is an induced subtrigraph of T that contains H as a weakly induced subgraph, contrary to the fact that T is $\mathcal{F}$-free. This proves (5.5.3). ■

(5.5.3) implies that three classes of strips do not occur in the strip-structure of $\mathcal{F}$-free claw-free trigraphs, more precisely:

(5.5.4) *Let (T, H, η) be a representation of some $\mathcal{F}$-free claw-free graph. Let $F \in E(H)$. Then, the strip of (H, η) at F is not isomorphic to a member of $\mathcal{Z}_5 \cup \mathcal{Z}_{12} \cup \mathcal{Z}_{14}$.*

Proof. Suppose that the strip of (H, η) at F is isomorphic to a member $(J, Z) \in \mathcal{Z}_5$. For $i \in [6]$, let v_i be as in the definition of $\mathcal{Z}_5$. Then, v_1-v_2-$\cdots$-v_6-v_1 is a weakly induced cycle of length six in J, contrary to (5.5.3). Next, suppose that $(J, Z) \in \mathcal{Z}_{12}$. Let $v_1, v_2, \dots, v_9, X$ be as in the definition of $\mathcal{Z}_{12}$. Let $j \in \{3, 4\}$ be largest such that v_2 is adjacent to v_j and let $k \in \{5, 6\}$ be smallest such that v_7 is adjacent to v_k. Such j, k exist by the fact that v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$ and v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$. But now v_1-v_2-v_j-v_k-v_7-v_8-v_1 is a weakly induced cycle of length six in J, contrary to (5.5.3). Finally, suppose that the strip of (H, η) at F is isomorphic to a member $(J, Z) \in \mathcal{Z}_{14}$. Let H', T', v_0, v_1, v_2, v_3 be as in the definition of $\mathcal{Z}_{14}$. Let $N = V(H') \setminus \{v_0, v_1, v_2, v_3\}$. Because $\deg(v_i) \geq 3$, for $i = 1, 2, 3$, there exist p_1, p_2, p_3 such that p_1, p_2 are complete to $\{v_1, v_2, v_3\}$ and p_3 is complete to $\{v_2, v_3\}$. Now v_1-p_1-v_2-p_3-v_3-p_2-v_1 is a cycle of length six in H'. Hence, T' has an weakly induced cycle of length six. Thus, J has a weakly induced cycle of length six, contrary to (5.5.3). ∎

Let T be a nonbasic claw-free trigraph and let (H, η) be a proper strip-structure for T. Let $F \in E(H)$ and let $\{u, v\} = \bar{F}$. Let $\ell(F)$ denote the set of integers k such that there exists a k-vertex weakly induced path from a vertex in $\eta(F, u)$ to a vertex to $\eta(F, v)$ whose interior vertices lie in $\eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$. Notice that $\ell(F) = \emptyset$ for $F \in E(H)$ such that one of $\eta(F, u)$ or $\eta(F, v)$ is empty, where $\{u, v\} = \bar{F}$ (the strip of (H, η) at such an F is a thickening of a member of $\mathcal{Z}_5 \cup \mathcal{Z}_6 \cup \ldots \cup \mathcal{Z}_{15}$). For a set of edges $S \subseteq E(H)$, we define

$$\ell(S) = \left\{ \sum_{F \in S} x_F \;\middle|\; x_F \in \ell(F), F \in S \right\}.$$

To clarify, $\ell(S)$ is the set of numbers that can be obtained by choosing for each $F \in S$ a number $x_F \in \ell(F)$ and taking the sum of these numbers $\{x_F\}_{F \in S}$. Notice that $1 \in \ell(F)$ if and only if $\eta(F, u) \cap \eta(F) \neq \emptyset$. Hence, $1 \in \ell(F)$ if and only if $\ell(F) = \{1\}$. A *cycle C in H* is a subgraph of H on vertex set $\{c_1, c_2, \dots, c_k\}$, with $k \geq 2$, and edge set $\{F_1, F_2, \dots, F_k\}$ such that $\bar{F}_i = \{c_i, c_{i+1}\}$ (with subscript modulo k). Notice that, by property (3) of the definition of a proper strip-structure, it follows that, for every cycle C in H, $\ell(F) \neq \emptyset$ for all $F \in E(C)$ and, thus, $\ell(E(C)) \neq \emptyset$. The following lemma deals with the possible values of $\ell(E(C))$ for cycles C in H.

(5.5.5) *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph. Let C be a cycle in H. Then, $z \in \{3, 4, 5, 7\}$ and $z \geq |E(C)|$ for all $z \in \ell(E(C))$.*

Proof. Suppose for a contradiction that there exists $z \in \ell(E(C)) \setminus \{3, 4, 5, 7\}$. Assume first that $z = 2$. Then it follows that $|E(C)| = 2$ and hence that the strips corresponding to the edges of C are spots. Let $F \in E(C)$. Clearly, T is a thickening of $T \setminus \eta(F)$, which contradicts the fact that (T, H, η) is an optimal representation. Hence, $z = 6$ or $z \geq 8$. Now, write $C = c_1$-c_2-$\ldots$-c_k-c_1 with $k = |E(C)|$ and such that, for all $i \in [k]$, there exists $F_i \in E(C)$ with $\bar{F}_i = \{c_i, c_{i+1}\}$ (subscripts modulo k). For $i \in [k]$, let $x_i \in \ell(F_i)$ be such that $z = \sum_{i \in [k]} x_i$ and let P_i be a weakly induced path from a vertex in $\eta(F_i, c_i)$ to a vertex in $\eta(F_i, c_{i+1})$ with $|V(P_i)| = x_i$. Now, P_1-P_2-$\ldots$-P_k-P_1 is a weakly induced cycle of length z, a contradiction. This proves (5.5.5). ∎

We need another lemma. For a trigraph T and a set $X \subseteq V(T)$, we say that $y \in V(T) \setminus X$ *is mixed on* X if y is not strongly complete or strongly anticomplete to X. We say that a set $Y \subseteq V(T) \setminus X$ *is mixed on* X if some vertex in Y is mixed on X.

(5.5.6) *Let T be a claw-free trigraph, and let $A, B, C \subseteq V(T)$ be disjoint nonempty sets in T such that A is strongly anticomplete to B, and C is a clique. Then, either at most one of A, B is mixed on C, or there exists a weakly induced 4-vertex path P with one endpoint in A and the other in B, and $V(P^*) \subseteq C$.*

Proof. Clearly, if $|C| = 1$, then it follows immediately from the fact that no vertex is incident with two semiedges that at most one of A, B is mixed on C. So we may assume that $|C| \geq 2$. We may assume that there exist $a \in A$ and $b \in B$ that are mixed on C. If a is complete to C, then let $X \subseteq C$ be the set of strong neighbors of a in C and let $Y \subseteq C$ be the set of antineighbors of a in C. If a is not complete to C, then let $X \subseteq C$ be the set of neighbors of a in C and let $Y \subseteq C$ be the set of strong antineighbors of a in C. Observe that $C = X \cup Y$ and, because $|C| \geq 2$ and a is mixed on C, X and Y are nonempty. If b has both an antineighbor $x \in X$ and a neighbor in $y \in Y$, then $P = a$-x-y-b is a weakly induced 4-vertex path with one endpoint in A and the other in B, and $|V(P^*)| \subseteq C$. Thus, we may assume that b is either strongly complete to X or strongly anticomplete to Y. Next, if b has both a neighbor $x' \in X$ and an antineighbor $y' \in Y$, then x' is complete to the triad $\{a, y', b\}$, contrary to (5.2.2). It follows that if b is strongly complete to X, then b is strongly complete to Y and, thus, b is not mixed on C. So we may assume that b is strongly anticomplete to Y. But now, it follows that b is strongly anticomplete to X and, thus, b is not mixed on C. It follows that B is not mixed on C, thereby proving (5.5.6). ■

This lemma allows us to rule out strips in which all weakly induced paths have the same length $k \geq 3$:

(5.5.7) *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Then, there exists no $F \in E(H)$ such that $\ell(F) = \{k\}$ for some $k \geq 3$.*

Proof. Assume for a contradiction that there exists $F \in E(H)$ such that $\ell(F) = \{k\}$ for some $k \geq 3$. Let (J, Z) be the strip of (H, η) at F. Let $\{u, v\} = \bar{F}$, $A = \eta(F, u)$, $B = \eta(F, v)$, and $C = \eta(F) \setminus (A \cup B)$. It follows from the fact that $1, 2 \notin \ell(F)$ that A and B are disjoint and A is strongly anticomplete to B.

Define the following sets. Let $N_0 = \{z_1\}$ and $N_{k+1} = \{z_2\}$, where z_1 is the unique vertex in Z that is strongly complete to A and z_2 is the unique vertex in Z that is strongly complete to B. Let $N_1 = A$ and $N_k = B$, and let $N_2, \dots, N_{k-1}$ be such that N_i is strongly anticomplete to N_j if $i < j - 1$, and N_i and N_{i+1} are linked. We may choose $N_2, \dots, N_{k-1}$ with maximal union and, since there exists a weakly induced path from a vertex in $N_1 = A$ to a vertex in $N_k = B$, $|N_i| \geq 1$ for all $i \in [k]$.

(i) *Let $x \in \eta(F) \setminus (N_1 \cup \dots \cup N_k)$. Then, there exists $i \in [k-1]$ such that x has a neighbor in N_i and in N_{i+1} and x is anticomplete to N_j with $j \neq i, i+1$.*

Let i be smallest such that x has a neighbor in N_i, say y, and let j be largest such that x has a neighbor in N_j. Clearly, since Z is strongly anticomplete to C, it follows that $1 \leq i \leq j \leq k$. First suppose that $i = j$. Then y has a neighbor $y_1 \in N_{i-1}$ and a neighbor $y_2 \in N_{i+1}$. But now, y is complete to the triad $\{x, y_1, y_2\}$, contrary to (5.2.2). Thus, $i \neq j$. If $|i - j| = 1$, then the lemma holds. Next, suppose that $|i - j| = 2$. Then, adding x to N_{i+1} contradicts the maximality of $N_1 \cup \ldots \cup N_k$. Thus, $|i - j| \geq 3$. But now, let P_1 be a weakly induced i-vertex path from a vertex in N_1 to a vertex in N_i, and let P_2 be a $(k - j)$-vertex path from a vertex in N_j to a vertex in B. Then, P_1-x-P_2 is a weakly induced path from a vertex in A to a vertex in B that has less that k vertices, a contradiction. This proves **(i)**. □

Next, for $i = 0, 1, \ldots, k$, let $M_{i,i+1} \subseteq \eta(F) \setminus (N_1 \cup \cdots \cup N_k)$ be the set of vertices with a neighbor in both N_i and N_{i+1}. It follows from **(i)** that $\eta(F) = \left(\bigcup_{i=1}^{k} N_i\right) \cup \left(\bigcup_{i=1}^{k-1} M_{i,i+1}\right)$. Also observe that $M_{0,1} = M_{k,k+1} = \emptyset$.

(ii) *For distinct $i, j \in [k-1]$, $M_{i,i+1}$ is strongly anticomplete to $M_{j,j+1}$.*

Suppose that $x \in M_{i,i+1}$ is adjacent to $y \in M_{j,j+1}$ for distinct $i, j \in [k-1]$. From the symmetry, we may assume that $i < j$. Now, let P_1 be a weakly induced i-vertex path from a vertex in N_1 to a vertex in N_i, and let P_2 be a $(k-j-1)$-vertex path from a vertex in N_{j+1} to a vertex in B. Then, P_1-x-y-P_2 is a weakly induced path from a vertex in A to a vertex in B that has $k' \neq k$ vertices, a contradiction. □

(iii) *For $i \in [k-1]$, $N_i \cup M_{i-1,i}$ is a strong clique.*

Since $|\bar{F}| = 2$, it follows from the definition of a proper strip-structure and (5.5.4) that the strip (J, Z) of (H, η) at F is isomorphic to a member of $\mathcal{Z}_l$ for some $l \in [4]$. If $l = 2, 3, 4$, then it follows immediately from the definition of the respective strips that C is a strong clique. So we may assume that $l = 1$. Thus, J is a linear interval trigraph. Thus, there exists a linear ordering $(\leq, V(J))$ such that for all adjacent $x, y \in V(J)$ and $z \in V(J)$, $x < z \leq y$ implies that z is strongly adjacent to x and y. We may assume that for every $x, y \in V(J)$, either $x > y$ or $x < y$. We prove a stronger statement:

($*$) *For $i \in [k-1]$, $N_i \cup M_{i-1,i}$ is a strong clique and $v_{i-1} < v_i$ for all adjacent $v_{i-1} \in N_{i-1}$, $v_i \in N_i \cup M_{i-1,i}$.*

We prove ($*$) by induction on i. First consider the case $i = 1$. $N_1 \cup M_{0,1}$ is a strong clique because $N_1 \cup M_{0,1} = A$, and it follows from our assumptions that $v_0 < v_1$ for all $v_0 \in N_0$ and $v_1 \in N_1 \cup M_{0,1}$. So let $i \geq 2$. We first claim that $v_{i-1} < v_i$ for all adjacent $v_{i-1} \in N_{i-1}$ and $v_i \in N_i \cup M_{i-1,i}$. For let $v_{i-1} \in N_{i-1}$ and $v_i \in N_i \cup M_{i-1,i}$ be adjacent. It follows from the definitions of N_{i-1}, N_i, and $M_{i-1,i}$ that v_{i-1} has a neighbor $v_{i-2} \in N_{i-2}$, and v_i is strongly antiadjacent to v_{i-2}. Inductively, $v_{i-2} < v_{i-1}$. Then it follows from the definition of a linear interval trigraph that $v_i > v_{i-1}$, as required.

Now suppose that $N_i \cup M_{i-1,i}$ is not a strong clique. Then there exist antiadjacent $x_1, x_2 \in N_i \cup M_{i-1,i}$. By the definition of N_i and $M_{i-1,i}$, x_1 and x_2 have neighbors $y_1, y_2 \in N_{i-1}$, and

y_1, y_2 have neighbors $z_1, z_2 \in N_{i-2}$, where possibly $z_1 = z_2$. Inductively, y_1 and y_2 are strongly adjacent. Since T is claw-free, it follows that both y_1, y_2 are not complete to $\{x_1, x_2\}$. Thus, $y_1 \neq y_2$, y_1 is strongly antiadjacent to x_2 and y_2 is strongly antiadjacent to x_1. It follows from the previous argument that $x_1 > y_1$ and $x_2 > y_2$. From the symmetry between x_1 and x_2, we may assume that $x_1 > x_2$. If $y_1 > x_2$, then the fact that $y_1 > x_2 > y_2$ and y_1 is adjacent to y_2 implies that x_2 is adjacent to y_1, a contradiction. Hence, $y_1 < x_2$. Now, $y_1 < x_2 < x_1$ and the fact that y_1 and x_1 are adjacent imply that x_2 is strongly adjacent to both y_1 and x_1, a contradiction. Thus, N_i is a strong clique. This proves **(iii)**. □

It follows from **(iii)** that, for $i \in [k-1]$, $N_i \cup M_{i-1,i}$ is a strong clique. From the symmetry, it follows that for $i \in [k] \setminus \{1\}$, $N_i \cup M_{i,i+1}$ is a strong clique. Thus, for $i \in [k-1]$, $M_{i,i+1}$ is strongly complete to $N_i \cup N_{i+1}$. Since T is claw-free, it follows that, for $i \in [k-1]$, $M_{i,i+1}$ is a strong clique.

(iv) *If, for some $j \in [k]$, N_j is strongly complete to N_{j+1}, then (T, H, η) is not an optimal representation of G.*

Let $j \in [k]$ be such that N_j is strongly complete to N_{j+1}. Construct a new strip-structure (H', η') for T from (H, η) as follows. First add to H' two new vertices w_1, w_2. Next, replace F by two new edges F_1, F_2 such that $\bar{F}_1 = \{u, w_1\}$, $\bar{F}_2 = \{v, w_1\}$. Let $\eta'(F_1) = \bigcup_{i=1}^{j}(N_i \cup M_{i-1,i})$, $\eta'(F_1, u) = A$, $\eta'(F_1, w_1) = N_j$, $\eta'(F_2) = \bigcup_{i=j+1}^{k}(N_i \cup M_{i,i+1})$, $\eta'(F_2, v) = B$, and $\eta'(F_2, w_1) = N_{j+1}$. If $M_{j,j+1} \neq \emptyset$, it follows from the fact that T is not a thickening of some other claw-free graph that $|M_{j,j+1}| = 1$; now add to H' an edge F_3 with $\bar{F}_3 = \{w_1, w_2\}$ $\eta'(F_3) = \eta'(F_3, w_1) = \eta'(F_3, w_2) = \{z\}$, where z is the unique vertex in $M_{j,j+1}$. Then, the strip of (H', η') at F_1, F_2 is isomorphic to a member of $\mathcal{Z}_1$, and, if $M_{j,j+1} \neq \emptyset$, the strip of (H', η') at F_3 is a spot. Thus, (T, H', η') is representation of G that satisfies $|E(H')| > |E(H)|$ and therefore, (T, H, η) is not an optimal representation, a contradiction. This proves **(iv)**. □

It follows from (5.5.6) that either at most one of N_1, N_3 is mixed on N_2, or there exists a weakly induced 4-vertex path $P = p_1\text{-}p_2\text{-}p_3\text{-}p_4$ with $p_1 \in A$, $p_2, p_3 \in N_2$, and $p_4 \in N_3$. If such P exists, then clearly, this path may be extended to obtain a $(k+1)$-vertex path from p_1 to a vertex in B, a contradiction. Thus, it follows that at least one of N_1, N_3 is not mixed on N_2. Since N_i and N_{i+1} are linked, it follows that at least one of N_1, N_3 is strongly complete to N_2, and thus the lemma holds by **(iv)**. This proves (5.5.7). ■

The previous lemma deals with strips in which all weakly induced paths have the same length $k \geq 3$. A question is: what happens when all weakly induced paths have length two? The next lemma deals with this case when such a strip is part of a long cycle.

(5.5.8) *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Let C be a cycle in H. If there exists $F \in E(C)$ such that $\ell(F) \in \{\{2\}, \{2,4\}\}$, then $\ell(E(C \setminus F)) \cap \{3,5\} = \emptyset$.*

Proof. Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Let C be a cycle in H and let $F \in E(C)$ be such that $\ell(F) \in \{\{2\}, \{2,4\}\}$. Let $\{u, v\} = \bar{F}$ and let $A' = \eta(F, u)$,

$B' = \eta(F, v)$, and $D' = \eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$. We start with the following claim:

(i) *D' is a strong clique.*

Since $|\bar{F}| = 2$, it follows from the definition of a proper strip-structure and (5.5.4) that the strip (J, Z) of (H, η) at F is isomorphic to a member of $\mathcal{Z}_l$ for some $l \in [4]$. If $l = 2, 3, 4$, then it follows immediately from the definition of the respective strips that D' is a strong clique. So we may assume that $l = 1$. Thus, J is a linear interval trigraph. Since $2 \in \ell(F)$, there exists adjacent $a \in A'$ and $b \in B'$. Now, it follows from the definition of a linear interval trigraph that D' is a strong clique. This proves **(i)**. □

We need to consider the graph G. Recall that G is a graphic thickening of T. For $u \in V(T)$, let X_u be the clique in G that corresponds to u. Let $A = \bigcup\{X_v \mid v \in A'\}$, and define B and D analogously.

(ii) *No vertex in D has nonadjacent neighbors $a \in A$ and $b \in B$.*

If $d \in D$ is adjacent to some nonadjacent $a \in A$ and $b \in B$, then a-d-b is an induced path that implies that $3 \in \ell(F)$, a contradiction. This proves **(ii)**. □

Assume for a contradiction that there exists $m \in \ell(E(C) \setminus \{F\})$ with $m \in \{3, 5\}$. It follows from the definition of a strip-structure that there exists a path p_1-p_2-$\dots$-p_m in G such that p_1 is complete to B, p_m is complete to A, $V(P^*)$ is anticomplete to $A \cup B$, and $V(P)$ is anticomplete to D. Let $A_0, A_1, A_2, \dots, A_k \subseteq A$ and $B_0, B_1, B_2, \dots, B_k \subseteq B$ be disjoint sets of vertices such that

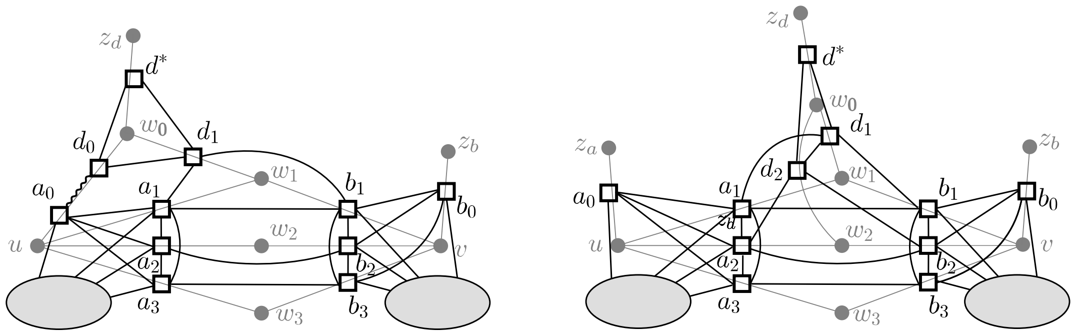
- for $0 \leq i, j \leq k$, $i \neq j$, A_i is anticomplete to B_j;
- A_0 is anticomplete to B_0;
- for $i \in [k]$, $|A_i| \geq 1$, $|B_i| \geq 1$, and A_i is complete to B_i.

We may choose these sets such that k is maximal and, subject to that, such that their union is maximal. Notice that we allow A_0 and B_0 to be empty, but the sets A_i, B_i, $i \in [k]$, are nonempty. Notice also that, because $2 \in \ell(F)$, $k \geq 1$.

(iii) *$A = \bigcup_{i=0}^k A_i$ and $B = \bigcup_{i=0}^k B_i$.*

Suppose not. From the symmetry, we may assume that there exists $a \in A \setminus \bigcup_{i=0}^k A_i$. Since $\bigcup_{i=0}^k A_i$ is maximal, it follows that a has a neighbor b in $\bigcup_{i=0}^k B_i$. Clearly, by the maximality of k, a has no neighbor in B_0. Also, a is not adjacent to $b_i \in B_i, b_j \in B_j$ with $i \neq j$. Indeed, if a has neighbors $b_i \in B_i, b_j \in B_j$ with $i \neq j$, then let $a_i \in A_i$. Now, $G|\{p_1, p_2, \dots, p_m, a, b_i, a_i, b_j\}$ is isomorphic to $\mathcal{G}_1$ (if $m = 3$) or $\mathcal{G}_2$ (if $m = 5$), a contradiction. Thus, we may assume that $b \in B_1$ and a is anticomplete to $(\bigcup_{i=0}^k B_i) \setminus B_1$. By the maximality of A_1, a has a nonneighbor $b' \in B_1$. Let $a' \in A_1$. Now, $G|\{p_1, p_2, \dots, p_m, a', b, a, b'\}$ is isomorphic to $\mathcal{G}_1$ (if $m = 3$) or $\mathcal{G}_2$ (if $m = 5$), a contradiction. This proves **(iii)**. □

Next, we analyze how vertices in D attach to $A \cup B$:



(a) $D_0 \neq \emptyset$ (b) $D_0 = \emptyset$

Figure 5.4: *The construction of a larger strip-structure in (5.5.8). The gray vertices and edges represent the relevant submultigraph of H'. The black vertices and edges represent the relevant induced subtrigraph of T'. The gray ellipses represent the sets K_1 and K_2. The 'wiggly' edge represents a semiedge. The black vertices are drawn on top of the gray edges to indicate to which strip each black vertex belongs.*

(iv) *For $i \in [k]$, if $d \in D$ has a neighbor in $A_i \cup B_i$, then d is complete to $A_i \cup B_i$ and anticomplete to $A \cup B \setminus (A_i \cup B_i)$.*

From the symmetry, we may assume that $d \in D$ has a neighbor $a \in A_i$. Let $b \in B_i$. Recall that A_i is complete to B_i. Hence, a is complete to $\{b, d, p_m\}$. It follows from (5.2.2) that d is adjacent to b. Thus, d is complete to B_i and, by the same argument, d is complete to A_i. It follows from **(ii)** that d is anticomplete to $A_j \cup B_j$ for $j \in [k] \cup \{0\}$, $j \neq i$. This proves **(iv)**. □

(v) *There do not exist $d_1, d_2 \in D$ such that d_1 has a neighbor in A_0 and d_2 has a neighbor in B_0.*

Let $d_1 \in D$ have a neighbor $a_0 \in A_0$ and let $d_2 \in D$ have a neighbor $b_0 \in B_0$. It follows from **(ii)** and **(iv)** that d_1 is anticomplete to $(A \cup B) \setminus A_0$ and d_2 is anticomplete to $(A \cup B) \setminus B_0$. Let $a_1 \in A_1$, $b_1 \in B_1$. Then, a_0-d_1-d_2-b_0-b_1-a_1-a_0 is an induced cycle of length six, a contradiction. This proves **(v)**. □

By **(v)** and the symmetry, we may assume that D is anticomplete to B_0. For $i \in [k] \cup \{0\}$, let D_i be the vertices in D that have a neighbor in $A_i \cup B_i$. It follows from **(iv)** that the sets $D_0, D_1, \dots, D_k$ are disjoint and that, for $i \in [k]$, D_i is complete to $A_i \cup B_i$. It follows from **(ii)** that D_0 is anticomplete to B. Let $D^* = D \setminus (D_0 \cup D_1 \cup \dots \cup D_k)$. We need one more lemma:

(vi) *There are at most two values $i \in [k] \cup \{0\}$ such that $D_i \neq \emptyset$.*

Suppose that there are i, j, l with $0 \leq i < j < l \leq k$ such that D_i, D_j and D_l are nonempty. It follows that A_i, A_l, B_j, B_l are all nonempty. Let $a_i \in A_i$, $a_l \in A_l$, $d_i \in D_i$, $d_j \in D_j$, $b_j \in B_j$ and $b_l \in B_l$ such that the pairs a_i, d_i and a_l, d_l are adjacent. Then, a_i-d_i-d_j-b_j-b_l-a_l-a_i is an induced cycle of length six, a contradiction. This proves **(vi)**. □

We will construct a new representation (T'', H', η'); see Figure 5.4 for an illustration of the construction. First construct T' from $T \setminus \eta(F)$ as follows. Let

$$K_1 = \bigcup\{\eta(F', u) \mid F' \in E(H) \setminus \{F\}, u \in \bar{F}'\}, \text{ and}$$
$$K_2 = \bigcup\{\eta(F', v) \mid F' \in E(H) \setminus \{F\}, v \in \bar{F}'\}.$$

Add a strong clique of new vertices $\bar{A} = \{a_0, a_1, \dots, a_k\}$ such that $\bar{A}$ is strongly complete to K_1, add a strong clique of new vertices $\bar{B} = \{b_0, b_1, \dots, b_k\}$ such that $\bar{B}$ is strongly complete to K_2, and add a strong clique of new vertices $\bar{D} = \{d_0, d_1, \dots, d_k\}$. If $D^* \neq \emptyset$, then add a new vertex d^* that is strongly complete to $\bar{D}$. For $i \in [k]$, let $\{a_i, b_i, d_i\}$ be a strong clique, and let a_0 be semiadjacent to d_0. All other pairs are strongly antiadjacent. Let $X' \subseteq \{a_0, b_0\}$ be such that $a_0 \in X'$ if and only if $A_0 = \emptyset$ and $b_0 \in X'$ if and only if $B_0 = \emptyset$. Let $X = X' \cup \{d_i \mid D_i = \emptyset, i \in [k] \cup \{0\}\}$. Let $T'' = T' \setminus X$. Then, G is a graphic thickening of T''.

Next, construct (H', η') from (H, η) as follows. First, delete F. For $i \in [k]$, add new vertices w_i, and edges $F_{1,i}, F_{2,i}$ with $\bar{F}_{1,i} = \{u, w_i\}$ and $\bar{F}_{2,i} = \{v, w_i\}$, and let $\eta'(F_{1,i}) = \eta'(F_{1,i}, u) = \eta'(F_{1,i}, w_i) = \{a_i\}$ and $\eta'(F_{2,i}) = \eta'(F_{2,i}, v) = \eta'(F_{2,i}, w_i) = \{b_i\}$. If $B_0 \neq \emptyset$, then add a new vertex z_b and an edge F_b with $\bar{F}_b = \{v, z_b\}$ and $\eta'(F_b) = \eta'(F_b, v) = \eta'(F_b, z_b) = \{b_0\}$. If $D = \emptyset$, then the construction of (T'', H', η') is complete. So from now on assume that $D \neq \emptyset$. Add a new vertex w_0. If $D^* \neq \emptyset$, then add a new vertex z_d and an edge F_d with $\bar{F}_d = \{w_0, z_d\}$ and $\eta'(F_d) = \eta'(F_d, w_0) = \eta'(F_d, z_d) = \{d^*\}$.

Now, there are two cases, depending on whether D_0 is empty or not. First suppose that $D_0 \neq \emptyset$. It follows from **(vi)** and the symmetry that we may assume that $D_i = \emptyset$ for all $i \in [k] \setminus \{1\}$. Add to H' a new edge F_0 with $\bar{F}_0 = \{u, w_0\}$, and $\eta'(F_0, u) = \{a_0\}$, $\eta'(F_0, w_0) = \{d_0\}$, and $\eta'(F_0) = \{a_0, d_0\}$. Notice that the strip of (H', η') at F_0 is a member of $\mathcal{Z}_1$. If $D_1 \neq \emptyset$, then add a new edge F_1 with $\bar{F}_1 = \{w_0, w_1\}$, $\eta'(F_1) = \eta'(F_1, w_0) = \eta'(F_1, w_1) = \{d_1\}$. This finishes the construction of (T'', H', η') when $D_0 \neq \emptyset$. (see Figure 5.4a)

Next, suppose that $D_0 = \emptyset$. It follows from **(vi)** and the symmetry that we may assume that $D_i = \emptyset$ for all $i \in [k] \setminus \{1, 2\}$. Since $D \neq \emptyset$, we may also assume that $D_1 \neq \emptyset$. For $i = 1, 2$, if $D_i \neq \emptyset$, then add a new edge F_i with $\bar{F}_i = \{w_i, w_0\}$, $\eta'(F_i) = \eta'(F_i, w_i) = \eta'(F_i, w_0) = \{d_i\}$. If $A_0 \neq \emptyset$, then add a new vertex z_a and an edge F_a with $\bar{F}_a = \{u, z_a\}$ and $\eta'(F_a) = \eta'(F_a, u) = \eta'(F_a, z_a) = \{a_0\}$. This finishes the construction of (T'', H', η'). (see Figure 5.4b)

Now G is a graphic thickening of T'', T'' is not a thickening of any other claw-free trigraph, (H', η') is a proper strip-structure for T'', and $|E(H')| > |E(H)|$, contrary to the fact that (T, H, η) is an optimal representation for G. This proves (5.5.8). ■

(5.5.9) *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Let C be a cycle in H and let $F \in E(C)$ be such that $\ell(E(C \setminus F)) \cap \{3, 4, 5, 6\} \neq \emptyset$. Then, the strip of (H, η) at F is a spot.*

Proof. We may assume that $\ell(F) \neq \{1\}$. If $6 \in \ell(E(C \setminus F))$, then it follows from (5.5.5) that $\ell(F) = \{1\}$, contrary to our assumption. If $5 \in \ell(E(C \setminus F))$, then it follows from (5.5.5) that

$\ell(F) = \{2\}$, contrary to (5.5.8). If $4 \in \ell(E(C \setminus F))$, then, since $\ell(F) \neq \{1\}$, it follows from (5.5.5) that $\ell(F) = \{3\}$, contrary to (5.5.7). Thus, we may assume that $3 \in \ell(E(C \setminus F))$. It follows from (5.5.5) that $\ell(F) \subseteq \{2,4\}$. It follows from (5.5.8) that $\ell(F) \neq \{2\}$ and $\ell(F) \neq \{2,4\}$. Thus, $\ell(F) = \{4\}$. But this contradicts (5.5.7). This proves (5.5.9). ■

Another useful corollary is the following description of possible strips in optimal representations:

(5.5.10) *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Let $F \in E(H)$ with $|\bar{F}| = 2$ and let $\{u, v\} = \bar{F}$. Then either*

(a) *the strip of (H, η) at F is a spot, or*

(b) *$\eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$ is a strong clique and $z \leq 4$ for all $z \in \ell(F)$, or*

(c) *the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_1$, $2 \notin \ell(F)$, and there exists $z \in \ell(F)$ with $z \geq 4$.*

Proof. Let (J, Z) be the strip of (H, η) at F. Since $|\bar{F}| = 2$, it follows from (5.5.4) that (J, Z) is isomorphic to a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$. If (J, Z) is isomorphic to a member of $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$, then it follows from the definition of the respective strips that $\eta(F) \setminus (\eta(F, u) \cap \eta(F, v))$ is a strong clique, and hence outcome (b) holds (the fact that $z \leq 4$ for all $z \in \ell(F)$ follows immediately). Therefore, we may assume that (J, Z) is isomorphic to a member of $\mathcal{Z}_1$, and thus J is a linear interval trigraph. Moreover, we may assume that $\ell(F) \subseteq \{2,3\}$, because otherwise either outcome (a) or (c) holds. Let $A = \eta(F, u)$, $B = \eta(F, v)$, and $C = \eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$. If $2 \in \ell(F)$, then there exist adjacent $a \in A$ and $b \in B$, and hence it follows from the definition of a linear interval trigraph that C is a strong clique and thus (b) holds. So we may assume that $\ell(F) = \{3\}$. But this contradicts (5.5.7). This proves (5.5.10). ■

5.5.2 The structure of the blocks of the multigraph in an optimal representation

Let T be a connected claw-free trigraph that admits a nontrivial strip-structure (H, η) such that H is not 2-connected. Let B be a block of H and let Z be the cut-vertices of H in $V(B)$. Let D be the trigraph obtained from $T|\bigcup\{\eta(F) \mid F \in E(B)\}$ by adding, for every $z \in Z$, a vertex y that is complete to $\bigcup\{\eta(F, z) \mid F \in E(B)\}$. Let Y be the vertices added in that way. We call (D, Y) the *strip-block of (H, η) at B.*

(5.5.11) *Let T be a connected $\mathcal{F}$-free claw-free trigraph that admits a nontrivial strip-structure (H, η). Let $(B_1, B_2, \dots, B_q)$ be the block decomposition of H. Then, at most one of $B_1, B_2, \dots, B_q$ contains a cycle of length at least five.*

Proof. Suppose that for distinct $i \in [2]$, B_i contains a cycle of length $k_i \geq 5$. It follows from the definition of a proper strip-structure that for $i \in [2]$ the strip-block (D_i, X_i) of (H, η) at B_i contains a weakly induced cycle C_i with $|V(C_i)| \in \{5, 7\}$. Because C_1 and C_2 are in different strip-blocks, it follows that $V(C_1) \cap V(C_2) = \emptyset$. Let $C_1 = c_1\text{-}\dots\text{-}c_{k_1}\text{-}c_1$ and $C_2 = c'_1\text{-}\dots\text{-}c'_{k_2}\text{-}c'_1$. Since T is connected, there exists a shortest weakly induced path $P = p_1\text{-}p_2\text{-}\dots\text{-}p_m$ from a vertex in $V(C_1)$ to a vertex in $V(C_2)$. We may assume that $p_1 = c_1$ and $p_m = c'_1$. First suppose that $m = 2$. Because c_1 is complete to $\{c'_1, c_2, c_{k_1}\}$, it follows that c'_1 is adjacent to at least one of c_2, c_{k_1}. From the symmetry, we may assume that c'_1 is adjacent to c_2. Symmetrically, we may assume that c_1 is adjacent to c'_2. Since, by the definition of a strip-structure, $N(C_1) \cap V(C_2)$ and $N(C_2) \cap V(C_1)$ are strong cliques, it follows that c_1 is strongly anticomplete to $V(C_2) \setminus \{c'_1, c'_2\}$ and c'_1 is strongly anticomplete to $V(C_1) \setminus \{c_1, c_2\}$. If c_2 is antiadjacent to c'_2, then c'_1 is complete to the triad $\{c_2, c'_2, c'_{k_2}\}$, contrary to (5.2.2). Thus, c_2 is strongly adjacent to c'_2. Since $N(C_1) \cap V(C_2)$ and $N(C_2) \cap V(C_1)$ are strong cliques, it follows that $N(C_1) \cap V(C_2) = \{c'_1, c'_2\}$ and $N(C_2) \cap V(C_1) = \{c_1, c_2\}$. Thus, $T|(V(C_1) \cup V(C_2))$ is a weakly induced skipping rope, a contradiction. So we may assume that $m \geq 3$. Since P is shortest, it follows that $V(C_1) \cup V(P^*)$ is strongly anticomplete to $V(C_2)$ and $V(C_2) \cup V(P^*)$ is strongly anticomplete to $V(C_1)$. Because c_1 is complete to $\{p_2, c_2, c_{k_1}\}$, it follows from (5.2.2) that p_2 is adjacent to at least one of c_2, c_{k_1}. We may assume that p_2 is adjacent to c_2. Next, if p_2 is complete to antiadjacent $c, c' \in V(C_1)$, then p_2 is complete to the triad $\{p_3, c, c'\}$, contrary to (5.2.2). Hence, it follows that p_2 is strongly anticomplete to $V(C_1) \setminus \{c_1, c_2\}$. Symmetrically, we may assume that p_m is complete to $\{c'_1, c'_2\}$ and strongly anticomplete to $V(C_2) \setminus \{c'_1, c'_2\}$. But now, $T|(V(C_1) \cup V(C_2) \cup V(P))$ is a weakly induced skipping rope, a contradiction. This proves (5.5.11). ■

As the previous lemma suggests, when we describe the blocks, it is convenient to distinguish between blocks that contain a cycle of length at least five, and blocks that do not contain such a cycle. We start with the former case. In **[6]**, we implicitly proved the following result. For completeness we give the proof of it here.

(5.5.12) *Let H be a 2-connected simple graph with no cycle of length k with $k = 6$ or $k \geq 8$. Then, either every cycle in H has length at most 4, or H is isomorphic to a graph in $\mathcal{B}_1$.*

Proof. We use induction on $|E(H)|$. Let $F = f_1\text{-}f_2\text{-}\dots\text{-}f_k\text{-}f_1$ be a largest cycle in H. If $k \leq 4$, then the lemma holds. Thus, since H has no cycle of length six or of length eight or more, we may assume that $k \in \{5, 7\}$. We say that a vertex $x \in V(H) \setminus V(F)$ is a *clone for F* if, for some $i \in [k]$, $N(x) \cap V(F) = \{f_{i-1}, f_{i+1}\}$ (subscript modulo k). In this case we say that x is a clone *of type i*. We start with a number of claims:

(i) *Every vertex in $V(H) \setminus V(F)$ is a clone for F.*

Let $x \in (V(H) \setminus V(F))$. Since H is 2-connected, there exist two paths P_1 and P_2 from x to two distinct vertices of F, say f_i and f_j, respectively, such that $V(P_1) \cap V(P_2) = \{x\}$. From the symmetry, we may assume that $i = 1$ and $j > k/2$. First assume that $|E(P_1)| + |E(P_2)| \geq 3$. Now $f_1\text{-}P_1^*\text{-}x\text{-}P_2^*\text{-}f_j\text{-}f_{j-1}\text{-}\cdots\text{-}f_2\text{-}f_1$ is a cycle of length $|E(P_1)| + |E(P_2)| + j - 1$ and $f_1\text{-}P_1^*\text{-}x\text{-}P_2^*\text{-}f_j\text{-}f_{j+1}\text{-}\cdots\text{-}f_k\text{-}f_1$ is a cycle of length $|E(P_1)| + |E(P_2)| + (k - j) - 1$. Thus, since H has no cycle of length six and by the maximality of F, we have

$$|E(P_1)| + |E(P_2)| + j - 1 \in [k] \setminus \{6\}, \text{ and } |E(P_1)| + |E(P_2)| + (k - j) - 1 \in [k] \setminus \{6\}.$$

It is straightforward to check that this system has no solution if $|E(P_1)| + |E(P_2)| \geq 3$. It follows that $|E(P_1)| + |E(P_2)| = 2$ and, therefore, $|E(P_1)| = |E(P_2)| = 1$. Thus, x has two neighbors in $V(F)$. If x has two consecutive neighbors in $V(F)$, say f_1, f_2, then f_1-x-f_2-f_3-$\cdots$-f_{k-1}-f_k-f_1 is a cycle of length $k+1$, contrary to the maximality of F. If $k = 5$, then, since x has at least two neighbors in $V(F)$, it follows that x is a clone for F. So we may assume that $k = 7$. Suppose that x is adjacent to f_p and f_{p+3} for some $p \in [7]$. From the symmetry, we may assume that $p = 1$. But now f_1-x-f_4-f_5-f_6-f_7-f_1 is a cycle of length six, a contradiction. From the symmetry, it follows that x has exactly two neighbors in F, say f_q and f_{q+2} for some $q \in [7]$. Hence, x is a clone for F. This proves **(i)**. $\square$

(ii) *If $x \in V(H) \setminus V(F)$ is a clone for F of type i, then no vertex in $V(H) \setminus V(F)$ is a clone of type $i+1$ (modulo k).*

From the symmetry, we may assume that x is a clone for F of type 1 and there exists $y \in V(H) \setminus V(F)$ that is a clone for F of type 2. Now, f_1-f_k-x-f_2-f_3-y-f_1 is a cycle of length six, a contradiction. This proves **(ii)**. $\square$

(iii) *$V(H) \setminus V(F)$ is a stable set.*

Suppose that $x, y \in V(H) \setminus V(F)$ are adjacent. From **(i)**, we may assume that x is a clone of type 1. From the symmetry and **(ii)**, we may assume that y is a clone of type 1, type 3, or, if $k = 7$, of type 4. First suppose that y is a clone of type 1. Then y-x-f_2-$\cdots$-f_k-y is a cycle of length $k+1$, contrary to the maximality of F. Next, suppose that y is a clone of type 3. Then, f_1-f_2-x-y-f_4-$\cdots$-f_k-f_1 is a cycle of length $k+1$, contrary to the maximality of F. Finally, suppose that $k = 7$ and y is a clone of type 4. Then f_2-f_3-f_4-f_5-y-x-f_2 is a cycle of length six, a contradiction. This proves **(iii)**. $\square$

Now suppose that there exists $x \in V(H) \setminus V(F)$. It follows from **(i)** that x is a clone for F. From the symmetry, we may assume that x is a clone of type 1. We claim that $\deg(f_1) = 2$. For suppose not. Then f_1 has a neighbor $y \in V(H) \setminus \{f_k, f_1, f_2\}$. First suppose that $y \in V(H) \setminus V(F)$. It follows from **(i)** that y is a clone of type 2 or type k, contrary to **(ii)**. Thus, it follows that $y = f_j$ for some $j \in \{3, \dots, k-1\}$. From the symmetry, we may assume that either $j = 3$, or $k = 7$ and $j = 4$. First assume that $j = 3$. Then x-f_2-f_1-f_3-$\dots$ f_k-x is a cycle of length $k+1$, a contradiction. So we may assume that $k = 7$ and $j = 4$. But now f_1-f_4-f_3-f_2-x-f_7-f_1 is a cycle of length six, a contradiction. This proves that $\deg(f_2) = 2$. Thus, H is obtained from $H \setminus \{x\}$ by cloning a vertex of degree two. Hence it follows from the induction hypothesis that H is isomorphic to a graph in $\mathcal{B}_1$ and therefore the lemma holds.

So we may assume that $V(H) = V(F)$. If $k = 5$, then H is isomorphic to a graph in $\mathcal{B}_1$ and the lemma holds. Therefore, we may assume that $k = 7$.

(iv) *Let $i \in [7]$. Then, f_i is nonadjacent to f_{i+2}.*

From the symmetry, we may assume that $i = 1$. If f_1 is adjacent to f_3, it follows that f_1-f_3-f_4-f_5-f_6-f_7-f_1 is a cycle of length six, a contradiction. This proves **(iv)**. $\square$

(v) *Let $i \in [7]$. If f_i is adjacent to f_{i+3}, then f_{i+5} is anticomplete to $\{f_{i+1}, f_{i+2}\}$.*

From the symmetry, we may assume that $i = 1$. Suppose that f_1 is adjacent to f_4. If f_6 is adjacent to f_2, then it follows that f_1-f_4-f_3-f_2-f_6-f_7-f_1 is a cycle of length six, a contradiction. This proves that f_6 is nonadjacent to f_2 and, symmetrically, f_6 is nonadjacent to f_3. This proves **(v)**. □

If F is an induced cycle, then the lemma holds. Therefore, it follows from **(iv)** and the symmetry that we may assume that f_1 is adjacent to f_4. It follows from **(v)** that f_6 is anticomplete to $\{f_2, f_3\}$. First suppose that f_2 is nonadjacent to f_5 and f_3 is nonadjacent to f_7. Then, the only undetermined adjacencies are between the pairs f_4, f_7 and f_1, f_5. Hence, H is of the $\mathcal{B}_1$ type and the lemma holds. Therefore, we may assume from the symmetry that f_2 is adjacent to f_5. It follows from **(v)** that f_7 is anticomplete to $\{f_3, f_4\}$. Now the only undetermined adjacency is between f_1 and f_5. Thus, H is of the $\mathcal{B}_1$ type. This proves (5.5.12). ■

Lemma (5.5.12) deals with blocks that contain a long cycle. For blocks with no such cycle, we use the following result from **[44]**.

Theorem 5.5.13. (**[44]**) *Let G be a graph. Then, the following statements are equivalent:*

(1) *G does not contain any odd cycle of length at least 5.*

(2) *For every connected subgraph G' of G, either G' is isomorphic to K_4, or G' is a bipartite graph, or G' is isomorphic to $K_{2,t}^+$ for some $t \geq 1$, or G' has a cut-vertex.*

This allows us to prove the following structural description of blocks that do not contain cycles of length at least five.

(5.5.14) *Let H be a 2-connected graph with $|V(H)| \geq 2$ that contains no cycle of length five or longer. Then, H is isomorphic to a graph in $\mathcal{B}_2$.*

Proof. It follows from Theorem 5.5.13 that either H is isomorphic to K_4, or H is a bipartite graph, or H isomorphic to $K_{2,t}^+$ for some $t \geq 1$. If H is isomorphic to K_4, then H is of the $\mathcal{B}_2$ type. If H is isomorphic to $K_{2,t}^+$ for some $t \geq 1$, then H is either isomorphic to K_3 (if $t = 1$), or to $K_{2,t}^+$ with $t \geq 2$, both of which imply that H is of the $\mathcal{B}_2$ type. Therefore, we may assume that H is a bipartite graph. Let $V(H) = X \cup Y$ such that X and Y are stable sets. If $|X| \leq 1$, then $x \in X$ is a cut-vertex, a contradiction. From the symmetry, it follows that $|X| \geq 2$ and $|Y| \geq 2$. Now suppose that $x \in X$ is nonadjacent to $y \in Y$. Since H is 2-connected, it follows that there are two edge-disjoint paths P_1 and P_2 from x to y. Since x and y are nonadjacent and H is bipartite, it follows that $|E(P_1)| \geq 3$ and $|E(P_2)| \geq 3$. But now x-P_1^*-y-P_2^*-x is a cycle of length at least six, a contradiction. It follows that X is complete to Y. If $|X| \geq 3$ and $|Y| \geq 3$, then clearly, H contains a cycle of length six, a contradiction. Therefore, at least one of X, Y has size exactly 2. Hence, H is isomorphic to $K_{2,t}$ with $t = \max\{|X|, |Y|\}$ and H is of the $\mathcal{B}_2$ type. This proves (5.5.14). ■

This allows us to prove (5.5.1):

Proof of (5.5.1). Let G be a connected $\mathcal{F}$-free claw-free graph that is a graphic thickening of a trigraph T that admits a nontrivial strip-structure. It follows that G has an optimal representation (T, H, η). Property (iii) follows from the following claim:

$(*)$ *Let C be a cycle in H with $|E(C)| \geq 4$. Then, $\ell(F) = \{1\}$ for all $F \in E(C)$.*

Let $F \in E(C)$. Since $|E(C \setminus F)| \geq 3$, it follows from (5.5.5) that $z \geq 3$ for all $z \in \ell(E(C \setminus F))$. It follows from (5.5.5) that $z \leq 6$ for all $z \in \ell(E(C \setminus F))$. Since $\ell(E(C \setminus F))$ is nonempty, it follows that $\ell(E(C \setminus F) \cap \{3, 4, 5, 6\} \neq \emptyset$ and, thus, by (5.5.9), $\ell(F) = \{1\}$. □

By (5.5.3), T is $\mathcal{F}$-free. It follows from the fact that T is $\mathcal{F}$-free that H has no cycles of length six or of length at least eight. Let $B_1, B_2, \ldots, B_q$ be the block-decomposition of H. Consider B_i. We claim that B_i is either of the $\mathcal{B}_1$ type, or of the $\mathcal{B}_2$ type. If B_i contains a cycle of length at least five, then it follows from (5.5.12) that B_i is of the $\mathcal{B}_1$ type. So we may assume that B_i has no cycle of length at least five. Now, it follows from (5.5.14) applied to $U(B_i)$ that B_i is of the $\mathcal{B}_2$ type. This proves part (i). Finally, for part (ii), it follows from (5.5.11) and the fact that every block of the $\mathcal{B}_1$ type contains a cycle of length five or seven, that at most one block of H is of the $\mathcal{B}_1$ type. This proves (5.5.1). ■

5.6 Theorem 5.0.8 for $\mathcal{F}$-free nonbasic claw-free graphs with stability number at most three

Recall that, by (5.2.10), all $\mathcal{F}$-free claw-free trigraphs with stability number at most 2 are resolved. In this section, we deal with nonbasic $\mathcal{F}$-free claw-free trigraphs with stability number 3.

Let T be a trigraph. Suppose that V_1, V_2 is a partition of $V(T)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

(1) $A_i, V_i \setminus A_i \neq \emptyset$, for $i = 1, 2$,

(2) $A_1 \cup A_2$ is a strong clique, and

(3) $V_1 \setminus A_1$ is strongly anticomplete to V_2, and V_1 is strongly anticomplete to $V_2 \setminus A_2$.

In these circumstances, we say that (V_1, V_2) is a *1-join*.

Next, suppose that V_0, V_1, V_2 are disjoint subsets with union $V(T)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

(1) $V_0 \cup A_1 \cup A_2$ and $V_0 \cup B_1 \cup B_2$ are strong cliques, and V_0 is strongly anticomplete to $V_i \setminus (A_i \cup B_i)$ for $i = 1, 2$,

(2) for $i = 1, 2$, $A_i \cap B_i = \emptyset$, and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all nonempty, and

(3) for all $v_1 \in V_1$ and $v_2 \in V_2$, either v_1 is strongly antiadjacent to v_2, or $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$.

We call the triple (V_0, V_1, V_2) a *generalized 2-join*.

Because the trigraphs that we are interested in are nonbasic, they admit a nontrivial strip-structure, and, since the stability number is 3, they admit either a 1-join or a generalized 2-join. The first lemma deals with clique cutsets (of which a 1-join is a special case).

(5.6.1) *Let T be an $\mathcal{F}$-free nonbasic claw-free trigraph with $\alpha(T) = 3$. If T has a clique cutset, then T is resolved.*

Proof. Let X be a clique cutset in T. Let $K_1, K_2, \dots, K_m$ be the connected components of $T \setminus X$. Since X is a clique cutset, $m \geq 2$. Because $\alpha(T) \leq 3$, it follows that for all i, j, at least one of K_i, K_j is a strong clique. Therefore, there exists i such that K_i is a strong clique. Now it follows from (5.2.8) applied to K_i and X that T is resolved. This proves (5.6.1). ■

The second lemma deals with generalized 2-joins.

(5.6.2) *Let T be an $\mathcal{F}$-free nonbasic claw-free trigraph with $\alpha(T) = 3$. Suppose that T admits a generalized 2-join. Then, T is resolved.*

Proof. For $i = 1, 2$, let V_i, A_i, B_i and V_0 be as in the definition of a generalized 2-join. Let $Q_i = V_i \setminus (A_i \cup B_i)$. In view of (5.6.1), we may assume that T has no clique cutset.

If, for some $i \in [2]$, A_i is strongly complete to B_i, then $A_i \cup B_i$ is a clique cutset in T, a contradiction. Hence, for $i \in [2]$, A_i is not strongly complete to B_i. Next, it follows from the fact that $\alpha(T) = 3$ and $Q_1, Q_2 \neq \emptyset$, that $\alpha(T|V_i) \leq 2$ for $i = 1, 2$. Let $\{i, j\} = \{1, 2\}$ and suppose that Q_i is not a strong clique. Since $\alpha(T) = 3$, it follows that V_j is a strong clique and hence that A_j is strongly complete to B_j, a contradiction. Thus, Q_1 and Q_2 are strong cliques.

Let $i \in [2]$. If $N(Q_i)$ is a strong clique, then $N(Q_i)$ is a clique cutset, a contradiction. It follows that there exist antiadjacent $a_i, b_i \in N(Q_i)$ and, because A_i and B_i are strong cliques, we may assume that $a_i \in A_i$ and $b_i \in B_i$. It follows that there exist $p_i, q_i \in Q_i$ (possibly equal) such that p_i is adjacent to a_i and q_i is adjacent to b_i. Since T has no weakly induced cycles of length six or of length at least 8, it follows that we may assume that $p_1 \neq q_1$, p_1 is strongly antiadjacent to b_1, q_1 is strongly antiadjacent to a_1, and $p_2 = q_2$. Since T has no weakly induced cycle of length six, it follows that A_2 is strongly anticomplete to B_2. Moreover, since $\alpha(T) = 3$, it follows from the fact that p_1 is antiadjacent to b_1 that Q_2 is strongly complete to A_2 and hence, from the symmetry, that Q_2 is strongly complete to B_2.

Let G be an $\mathcal{F}$-free graphic thickening of T. We claim that G is resolved. For $v \in V(T)$, let X_v be the clique in G corresponding to v. For $i \in [2]$, let $V_i' = \bigcup \{X_v \mid v \in V_i\}$ and define A_i', B_i', Q_i', and V_0' analogously. Observe that T contains a weakly induced cycle of length seven. Therefore, by (5.2.1) and the strong perfect graph theorem **[17]**, G is not perfect. Thus, if every maximal stable set in G has size three, then G satisfies condition (c) of the definition of a resolved graph and hence G is resolved. Clearly, no vertex is complete to all other vertices, so there is no maximal stable set of size one. So we may assume that there exists a maximal stable set $S = \{s_1, s_2\}$ of size two in G. If $S \cap V_2' = \emptyset$, then we may add any vertex from Q_2' to S to obtain a larger stable set, a contradiction. If $S \subseteq V_2'$, then we may add any vertex from Q_1' to S to obtain a larger stable set, a contradiction. It follows that $|S \cap V_2'| = 1$ and hence $|S \cap (V_0' \cup V_1')| = 1$. We may assume that $s_1 \in V_0' \cup V_1'$ and $s_2 \in V_2'$. If $s_1 \in V_0'$, then we may add any vertex from Q_1' to S to obtain a larger stable set, a contradiction. It follows that $s_1 \in V_1'$. We need the following observation:

(∗) *If $q_1' \in Q_1'$ has neighbors $a_1' \in A_1'$ and $b_1' \in B_1'$, then a_1' is adjacent to b_1'.*

Suppose not. Then, let $a_2' \in A_2'$, $b_2' \in B_2'$, and $q_2' \in Q_2'$ and observe that a_1'-q_1'-b_1'-b_2'-q_2'-a_2'-a_1' is an induced cycle of length six, a contradiction. This proves **(∗)**. □

First suppose that $s_1 \in Q_1'$. Since A_1' is not complete to B_1', there exist nonadjacent $a_1' \in A_1'$ and $b_1' \in B_1'$. It follows from **(∗)** that s_1 is not complete to $\{a_1', b_1'\}$. From the symmetry, we may assume that s_1 is nonadjacent to a_1'. It follows from the maximality of S that $s_2 \in A_2$. But now, we may add any vertex from B_2' is S to obtain a larger stable set, a contradiction. This proves that $s_1 \notin Q_1'$. Therefore, from the symmetry, we may assume that $s_1 \in A_1'$. The maximality of S implies that s_1 is complete to Q_1'. In particular, s_1 is complete to X_{p_1} and X_{q_1}. Since X_{q_1} is complete to X_{b_1}, it follows from **(∗)** that s_1 is complete to X_{b_1}. But now, s_1 is complete to the triad $\{a_2', b_1', p_1'\}$ with $a_2' \in A_2'$, $b_1' \in X_{b_1}$ and $p_1' \in X_{p_1}$, contrary to (5.2.2). This proves that G is resolved, which implies that T is resolved, thus proving (5.6.2). ■

This leads to the main result of this subsection:

(5.6.3) *Every nonbasic $\mathcal{F}$-free claw-free trigraph T with $\alpha(T) \leq 3$ is resolved.*

Proof. If $\alpha(T) \leq 2$, then it follows from (5.2.10) that T is resolved. Thus, we may assume that $\alpha(T) = 3$. Since T is nonbasic, T admits a proper strip-structure. In particular, T admits a 1-join or a generalized 2-join. Hence, it follows from (5.6.1), (5.6.2) that T is resolved. This proves (5.6.3). ■

5.7 Theorem 5.0.8 for $\mathcal{F}$-free nonbasic claw-free graphs

The goal of this section is to prove Theorem 5.0.8 for $\mathcal{F}$-free nonbasic claw-free graphs. To be precise we will prove the following:

(5.7.1) *Every connected $\mathcal{F}$-free nonbasic claw-free trigraph is resolved.*

We are now ready to prove that nonbasic $\mathcal{F}$-free claw-free graphs are resolved. In Section 5.6, we dealt with nonbasic trigraphs that have stability number at most three, so we may assume that our trigraphs have stability number at least four. In view of the definition of a (tri)graph being resolved, this means that we always look for dominant cliques. In Section 5.5, we gave a structure theorem for the multigraph H of an optimal representation (T, H, η) of an $\mathcal{F}$-free nonbasic claw-free trigraph and we stated this structure in terms of the block decomposition of H. After introducing a few more lemmas in Subsection 5.7.1, we will deal, in Subsection 5.7.2, with trigraphs for which the multigraph of an optimal representation is 2-connected. Then, in Subsection 5.7.3, we will deal with trigraphs whose multigraph in an optimal representation is not 2-connected.

5.7.1 Tools

We need a few more tools that help us conclude that graphs are resolved. We need the following result on clones of vertices of degree 2.

(5.7.2) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Suppose that there exist $x_1, x_2 \in V(H)$ with $\deg(x_1) = \deg(x_2) = 2$ and $N(x_1) = N(x_2)$, and $\ell(x_i w) = \{1\}$ for all $i \in \{1, 2\}$ and $w \in N(x_i)$. Then, G is resolved.*

Proof. Let $\{u, v\} = N(x_1) = N(x_2)$. Let E_1 be the set of edges in H incident with x_1. Let $K = \bigcup\{\eta(F, u) \mid F \in E_1\}$. We claim that K is a dominant clique in T. For suppose not. Then, there exists a stable set $S \subseteq V(T) \setminus K$ that covers K. For $i = 1, 2$, let $z_i \in \eta(ux_i)$. For $i \in \{1, 2\}$, since $z_i \notin S$ and S covers K, it follows that there exist $y_i \in S$ that is adjacent to z_i. It follows from the assumptions and the choice of K that $y_i \in \eta(vx_i)$. But now it follows that y_1 and y_2 are strongly adjacent, contrary to the fact that S is a stable set. This proves (5.7.2). ■

(5.7.3) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let $F \in E(H)$ and let $\{u, v\} = \bar{F}$. If $\ell(F) = \{2\}$, then either $\eta(F) = \eta(F, u) \cup \eta(F, v)$, or T is resolved.*

Proof. Let $A = \eta(F, u)$, $B = \eta(F, v)$, $C = \eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$. We may assume that $C \neq \emptyset$, because otherwise the lemma holds. Since $2 \in \ell(F)$, it follows from (5.5.10) that C is a strong clique. If $N(C)$ is a strong clique, then (5.2.8) applied to $N(C)$ and C implies that G is resolved, and the lemma holds. Thus, we may assume that $N(C)$ is not a strong clique. Therefore, since A, B are strong cliques and $N(C) \subseteq A \cup B$, there exist antiadjacent $a \in A \cap N(C)$, $b \in B \cap N(C)$ and a weakly induced path P from a to b with $V(P^*) \subseteq C$ and $|V(P)| \in \{3, 4\}$. But this implies that $|V(P)| \in \ell(F)$, a contradiction. This proves (5.7.3). ■

5.7.2 2-connected strip-structures

We start with trigraphs whose multigraph in the optimal representation is 2-connected.

(5.7.4) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation* *of G. If H is 2-connected, then G is resolved.*

Proof. In view of (5.6.3), we may assume that $\alpha(T) \geq 4$. It follows from (5.5.1) and the fact that H is 2-connected that H is either of the $\mathcal{B}_1$ type or of the $\mathcal{B}_2$ type.

First suppose that H is of the $\mathcal{B}_1$ type. Since every edge of H is in a cycle of length 4, 5, or 7, it follows from (5.5.1) that $\ell(F) = \{1\}$ for all $F \in E(H)$. If there exist $F_1, F_2 \in E(H)$ with $\bar{F}_1 = \bar{F}_2$ then $\{F_1, F_2\}$ is a cycle that contradicts (5.5.5). Thus, H has no parallel edges. It follows that T, regarded as a graph, is the line graph of H. If H contains nonadjacent clones of vertices of degree 2, then it follows from (5.7.2) that G is resolved. So we may assume that H contains no such clones, and thus $U(H)$ is isomorphic to a graph in $\mathcal{B}_1^*$. But now, it is straightforward to check that $\alpha(T) \leq 3$, a contradiction.

So we may assume that H is of the $\mathcal{B}_2$ type. It follows that $U(H)$ is either isomorphic to K_m for some $m \in \{2, 3, 4\}$, or to $K_{2,t}$ or $K_{2,t}^+$ for some $t \geq 2$. We prove the lemma by considering each case separately.

(i) *If $U(H)$ is isomorphic to K_2, then there is no F^* with $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$.*

Suppose that such F^* exists. Let u, v be the unique two vertices of H. It follows from the fact that (H, η) is nontrivial that $|E(H)| \geq 2$. Clearly, if all strips of (H, η) are spots, then $\alpha(T) = 1$, a contradiction. Thus, the strip of (H, η) at F^* is not a spot. First suppose that $\eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$ is a strong clique. Then, T is the union of three strong cliques $\bigcup\{\eta(F, u) \mid F \in E(H)\}$, $\bigcup\{\eta(F, v) \mid F \in E(H)\}$, and $\eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$, and thus $\alpha(T) \leq 3$, a contradiction. Thus, $\eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$ is not a strong clique. It follows from (5.5.10) that the strip of (H, η) at F^* is in $\mathcal{Z}_1$ and that $2 \notin \ell(F^*)$. Now, T is a long circular interval trigraph, a contradiction. This proves **(i)**. □

(ii) *If $U(H)$ is isomorphic to K_2, then G is resolved.*

It follows from the fact that (H, η) is nontrivial that $|E(H)| \geq 2$. Let z be maximum such that $z \in \ell(F^*)$ for some $F^* \in E(H)$. It follows from (5.5.5) that $z \leq 6$, and it follows from **(i)** that $z \geq 2$. Let $\{u, v\} = \bar{F}^* = V(H)$. Now there are five cases.

First suppose that $z = 6$. It follows that $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$, contrary to **(i)**. Next, suppose that $z = 5$. Let $F \in E(H) \setminus \{F^*\}$. It follows from (5.5.5) that $\ell(F) = \{2\}$, contrary to (5.5.8). Next, suppose that $z = 4$. It follows from (5.5.5) that $\ell(F) \in \{\{1\}, \{3\}\}$ for all $F \in E(H) \setminus \{F^*\}$. Since, by (5.5.7), no $F \in E(H) \setminus \{F^*\}$ satisfies $\ell(F) = \{3\}$, it follows that $\ell(F) \in \{1\}$ for all $F \in E(H) \setminus \{F^*\}$, contrary to **(i)**. Now, suppose that $z = 3$. It follows from (5.5.5) that either $\ell(F) = \{1\}$ or $\ell(F) = \{2\}$ for all $F \in E(H) \setminus \{F^*\}$. It follows from (5.5.8) that $\ell(F) \not\subseteq \{2, 4\}$ for all $F \in E(H) \setminus \{F^*\}$. Therefore, $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$, contrary to **(i)**. So we may assume that $z = 2$. It follows from (5.5.10) that for every $F \in E(H)$ with $\ell(F) = \{2\}$, $\eta(F) = \eta(F, u) \cup \eta(F, v)$. Hence, T is the union of two strong cliques, namely $\bigcup\{\eta(F, u) \mid F \in E(H)\}$ and $\bigcup\{\eta(F, v) \mid F \in E(H), \ell(F) = \{2\}\}$. Therefore, $\alpha(T) \leq 2$, a contradiction. This proves **(ii)**. □

(iii) *If $U(H)$ is isomorphic to K_3, then G is resolved.*

Let z be maximum such that $z \in \ell(F^*)$ for some $F^* \in E(H)$. It follows from (5.5.5) that $z \leq 5$. Let $V(H) = \{c_1, c_2, c_3\}$ such that $\bar{F}^* = \{c_1, c_2\}$. Now, there are five cases.

First suppose that $z = 5$. Then, by (5.5.5), $\ell(F) = \{1\}$ for all $F \in E(H)$ such that $\bar{F} \neq \{c_1, c_2\}$. If there exists $F \in E(H) \setminus \{F^*\}$ such that $\bar{F} = \{c_1, c_2\}$, then it follows from (5.5.5) that $\ell(F) = \{2\}$, contrary to (5.5.8). Thus, no such F exists. It follows from (5.5.10) that the strip of (H, η) at F^* is in $\mathcal{Z}_1$ and that $2 \notin \ell(F^*)$. But now, T is a long circular interval trigraph, a contradiction. Next, suppose that $z = 4$. Let $F_1, F_2 \in E(H)$ be such that $\bar{F}_1 = \{c_1, c_3\}$ and $\bar{F}_2 = \{c_2, c_3\}$. It follows from (5.5.5) that exactly one of F_1, F_2, say F', satisfies $\ell(F') = \{2\}$. But now consider $C = \{F^*, F_1, F_2\}$. It follows that $5 \in \ell(E(C) \setminus F')$, contrary to (5.5.8). Now, suppose that $z = 3$. It follows from (5.5.7) that $\ell(F^*) = \{2, 3\}$. Therefore, it follows from (5.5.8) that that $\ell(F) = \{1\}$ for all $F \in E(H)$ with $\bar{F} \neq \{c_1, c_2\}$. Moreover, it follows from (5.5.5) and (5.5.8) that $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$ with $\bar{F} = \{c_1, c_2\}$. It follows from (5.5.10) that $\eta(F^*) \setminus (\eta(F^*, c_1) \cup \eta(F^*, c_2))$ is a strong clique. Now, T is the union of three strong cliques $\bigcup\{\eta(F, c_1) \mid F \in E(H)\}$, $\bigcup\{\eta(F, c_2) \mid F \in E(H)\}$, and $\eta(F^*) \setminus (\eta(F^*, c_1) \cup \eta(F^*, c_2))$. Thus, $\alpha(T) \leq 3$, a contradiction. Next, suppose that $z = 2$. It follows that $\ell(F^*) = \{2\}$. Hence $\ell(F) = \{1\}$ for all $F \in E(H)$ with $\bar{F} \neq \bar{F}^*$. Indeed suppose that for some $F_1 \in E(H)$ with $\bar{F}_1 \neq \bar{F}^*$, we have $l(F_1) = \{2\}$. Then consider the cycle $C = \{F^*, F_1, F_2, \}$, where $F_2 \in E(H)$ and $\bar{F}_2 \neq \bar{F}^*$. Now it follows that $3 \in \ell(E(C) \setminus F_1)$, contrary to (5.5.8). It follows from (5.7.3) that for every $F \in E(H)$ with $\ell(F) = \{2\}$, $\eta(F) = \eta(F, c_1) \cup \eta(F, c_2)$. Hence, T is the union of two strong cliques $\bigcup\{\eta(F, c_1) \mid F \in E(H)\}$ and $\bigcup\{\eta(F, c_2) \mid F \in E(H)\}$. Thus, $\alpha(T) \leq 2$, a contradiction. Therefore, we may assume that $z = 1$. Now T is a strong clique and $\alpha(T) = 1$, a contradiction. This proves **(iii)**. □

(iv) *If $U(H)$ is isomorphic to K_4, then G is resolved.*

Since every edge of H is in a cycle of length four, (5.5.1) implies that $\ell(F) = \{1\}$ for all $F \in E(H)$. It follows that T, regarded as a graph, is the line graph of K_4. But now, $\alpha(T) \leq 2$, a contradiction. This proves **(iv)**. □

(v) *For $t \geq 2$, if $U(H)$ is isomorphic to $K_{2,t}$ or $K_{2,t}^+$, then G is resolved.*

Let Y and Z be such that Y is a stable set and Z satisfies $|Z| = 2$. Write $Y = \{y_1, \dots, y_t\}$ and $Z = \{z_1, z_2\}$. Let $E' \in E(H)$ be the set of edges $F \in E(H)$ with $\bar{F} = \{z_1, z_2\}$. Since every edge in $E(H) \setminus E'$ is in a cycle of length four, (5.5.1) implies that $\ell(F) = \{1\}$ for all $F \in E(H) \setminus E'$. But now, y_1 and y_2 are nonadjacent clones in H that satisfy the assumptions of (5.7.2) and therefore G is resolved by (5.7.2). This proves **(v)**. □

Thus, it follows from **(ii)**, **(iii)**, **(iv)**, and **(v)** that G is resolved. This proves (5.7.4). ■

5.7.3 Strip-structures that are not 2-connected

Let T be a connected nonbasic claw-free trigraph and let (T, H, η) be an optimal representation of T. We say that a block B of H is a *leaf-block* if B contains exactly one cut-vertex of H. In Figure 5.3, for example, the block labeled $K_{2,4}^{+}$ is a leaf-block. We call a strip-block that corresponds to a leaf-block in H a *leaf strip-block*.

Let G be an $\mathcal{F}$-free nonbasic claw-free trigraph and let (T, H, η) be an optimal representation of G. Let B be a leaf-block of H. Consider the strip-block (D, Y) of (H, η) at B. Because B is a leaf-block, there is a unique $y \in Y$. Construct the graph D' from $G|\bigcup\{X_v \mid v \in V(D)\}$ by adding a new vertex y' that is strongly complete to $Y' = \bigcup\{X_v \mid v \in N_D(y)\}$. Then, G contains D' as an induced subgraph. If D' contains no induced heft with end y', then (D, Y) is said to be *ordinary (with respect to G)*.

It turns out that if we consider two leaf strip-blocks of an $\mathcal{F}$-free claw-free trigraph T, then at least one them has to be ordinary with respect to a fixed $\mathcal{F}$-free thickening of T (we will prove this in (5.7.5)). In particular, since the multigraphs of the strip-structures that we are interested in at this point are not 2-connected, there exists at least one ordinary leaf strip-block. Our strategy for concluding that graphs with non-2-connected strip-structures are resolved is to consider such an ordinary leaf strip-block, and find a dominant clique contained in it.

We note that, in the definition of an ordinary leaf strip-block, it is necessary to refer to a specific graphic thickening, because in general the leaf strip-block that is ordinary depends on the graphic thickening. Consider, for example, Figure 5.5. The diagram on the left depicts an $\mathcal{F}$-free nonbasic claw-free trigraph T and the diagram on the right shows a graphic thickening G of T, where, for $i = 1, 2$, the vertices in V_i' correspond to the vertices in V_i. With respect to the graphic thickening G, the strip-block corresponding to the set V_2 in T is ordinary and the strip-block corresponding to the set V_1 in T is not ordinary. But by rotating the graphic thickening by 180 degrees, it is clear that with respect to a different graphic thickening, it is possible that the left hand side of the 1-join in T is ordinary. In fact, there are exactly two dominant cliques in G, namely $\{u_1, u_2, u_3\}$ and $\{w_1, w_2\}$, which shows that it is not possible to know where to find a dominant clique from the trigraph alone.

Tools

We need a few lemmas on ordinary leaf strip-blocks.

(5.7.5) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Suppose that B_1, B_2 are two distinct leaf-blocks of H. Then, the strip-block of (H, η) at at least one of B_1, B_2 is ordinary.*

Proof. Suppose that for $i = 1, 2$, the strip-block (D_i, Y_i) of (H, η) at B_i is not ordinary. Let D_i', y_i' be as in the definition of the ordinary strip-block (D_i, Y_i). Because B_i is not ordinary, it follows that D_i has an induced heft H_i with end y_i'. Because G is connected and B_1 and B_2 are leaf-blocks, it follows that there exists an induced path $P = p_1\text{-}p_2\text{-}\dots\text{-}p_k$, with $k \geq 2$, from a vertex in $N(y_1') \cap V(H_1)$ to a vertex in $N(y_2') \cap V(H_2)$, and $V(P^*)$ is disjoint from $V(D_1') \cup V(D_2')$. It follows from the definition

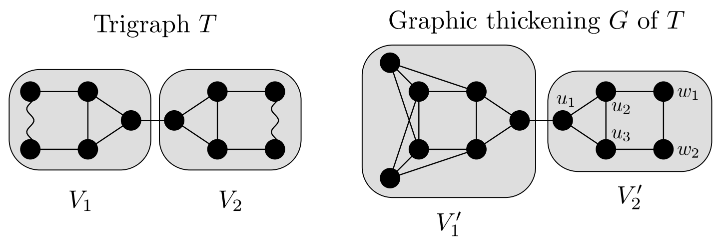


Figure 5.5: *An example of a trigraph T (left) for which it is not possible to determine from the trigraph alone which leaf strip-block is the ordinary block given by (5.7.5). The graph on the right shows a $\mathcal{F}$-free graphic thickening of T.*

of a strip-structure that p_2 is strongly complete to $N(y_1') \cap V(H_1)$ in G and p_{k-1} is strongly complete to $N(y_2') \cap V(H_2)$ in G. Now, $G|V(H_1) \cup V(H_2) \cup V(P)$ is a skipping rope, a contradiction. This proves (5.7.5). ■

We have the following useful properties of ordinary strip-blocks:

(5.7.6) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf-block of H and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. Then, all of the following hold:*

- (a) *D contains no weakly induced heft with its end in Y;*
- (b) *D contains no weakly induced cycle of length at least five;*
- (c) *B is of the $\mathcal{B}_2$ type;*
- (d) *for every cycle C in B, $\ell(E(C)) \subseteq \{3, 4\}$.*

Proof. Part (a) follows immediately from the definition of D, Y, and from (5.2.1). For part (b), suppose that D contains a weakly induced cycle $C = c_1\text{-}c_2\text{-}\dots\text{-}c_k\text{-}c_1$ of length $k \geq 5$. Since T is $\mathcal{F}$-free, it follows that $k \in \{5, 7\}$. Since every vertex in Y is simplicial in D, it follows that $Y \cap V(C) = \emptyset$. However, since D is connected, there exists a path P from a vertex $y \in Y$ to a vertex in $V(C)$ with interior in $V(D) \setminus Y$. From the symmetry, we may write $P = p_1\text{-}p_2\text{-}\dots\text{-}p_m$, where $m \geq 2$ and $p_1 = y$ and $p_m = c_1$. Since P is shortest, it follows that, for $1 \leq j < m-1$, p_j is anticomplete to $V(C)$. We first claim that p_{m-1} does not have two antiadjacent neighbors $c, c' \in V(C)$. For suppose it does. Since p_1 is a simplicial vertex, it follows that $m \geq 3$. But now, p_{m-1} is complete to the triad $\{c, c', p_{m-2}\}$, a contradiction. Thus, p_{m-1} does not have two antiadjacent neighbors $c, c' \in V(C)$. If p_{m-1} is anticomplete to $\{c_2, c_k\}$, then c_1 is complete to the triad $\{c_2, c_k, p_{m-1}\}$, a contradiction. Thus, since p_{m-1} is not complete to $\{c_2, c_k\}$, we may assume that p_{m-1} is strongly adjacent to c_2 and strongly antiadjacent to c_k. Since every vertex in $V(C) \setminus \{c_1, c_2\}$ is antiadjacent to one of c_1, c_2, it follows that p_{m-1} is strongly anticomplete to $V(C) \setminus \{c_1, c_2\}$. Now, $T|(V(P) \cup V(C))$ is a weakly induced heft with end $y \in Y$, a contradiction. This proves (b). Part (c) follows from part (b), (5.5.1), and the fact that if B is of the $\mathcal{B}_1$ type, then D contains a weakly induced cycle of length at least five. Part (d) follows immediately from part (b). This proves (5.7.6). ■

This lemma implies that some types of strips $\mathcal{Z}_i$ cannot occur in ordinary blocks:

(5.7.7) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf-block of H such that the strip-block of (H, η) at B is ordinary, and let $F \in E(B)$. Then, the strip of (H, η) at F is not isomorphic to a member of $\mathcal{Z}_4 \cup \mathcal{Z}_7 \cup \mathcal{Z}_8 \cup \mathcal{Z}_{13}$.*

Proof. Let (J, Z) be the strip of (H, η) at F. For notational convenience, we may assume that (J, Z) is a member of $\mathcal{Z}_4 \cup \mathcal{Z}_7 \cup \mathcal{Z}_8 \cup \mathcal{Z}_{13}$ (as opposed to isomorphic to a member of that family). We will go through the classes of strips one by one. It follows from (5.7.6) that J contains no weakly induced cycle of length five. First suppose that $(J, Z) \in \mathcal{Z}_4$. Let $T, a_1, a_2, c_1, b_2, b_1$ be as in the definition of $\mathcal{Z}_4$. Then, a_1-a_2-c_1-b_2-b_1-a_1 is a weakly induced cycle of length five in J, a contradiction. Thus, $(J, Z) \notin \mathcal{Z}_4$. Next, suppose that $(J, Z) \in \mathcal{Z}_7$. Let $H, H', h_1, \dots, h_5$ be as in the definition of $\mathcal{Z}_7$. Since h_1-h_2-$\cdots$-h_5-h_1 is a cycle of length five in H, it follows that J has an induced cycle of length five, contrary to (5.7.6). Now, suppose that $(J, Z) \in \mathcal{Z}_8$. Let $A, B, C, X, d_1, d_3, d_4$ be as in the definition of $\mathcal{Z}_8$. Because $A \setminus X$ is not strongly complete to $B \setminus X$, there exist antiadjacent $a \in A$ and $b \in B$. But now, d_1-a-d_3-d_4-b-d_1 is a weakly induced cycle of length five, a contradiction. Finally, suppose that $(J, Z) \in \mathcal{Z}_{13}$. Let T', L_1, L_2, L_3 be as in the definition of $\mathcal{Z}_{13}$. Then, $(T', V(T') \cap L_1, V(T') \cap L_2, V(T') \cap L_3)$ is a three-cliqued claw-free trigraph that belongs to the class $\mathcal{TC}_2$. It follows from (5.4.16) that T' is contains a semihole of length at least five. Since T' is an induced subtrigraph of J, it follows that J contains a semihole of length at least five, a contradiction. This proves (5.7.7). ■

The following lemma is a counterpart of (5.2.11) for ordinary strip-blocks.

(5.7.8) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf-block of H and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. Suppose that (K_1, K_2) is a homogeneous pair of cliques in D such that K_1 is not strongly complete and not strongly anticomplete to K_2, and $V(K_1) \cup V(K_2)$ is strongly anticomplete to Y. For $\{i, j\} = \{1, 2\}$, let $N_i = N(K_i) \setminus N[K_j]$. If some vertex in Y has neighbor in both N_1 and N_2, then G is resolved.*

Proof. Let $M = V(T) \setminus (N[K_1] \cup N[K_2])$. Let G be an $\mathcal{F}$-free graphic thickening of T. For $v \in V(T)$, let X_v denote the corresponding clique in G. Let $K_1' = \bigcup\{X_v \mid v \in K_1\}$ and define K_2', N_1', N_2', Y', M' analogously. Let $Z' = (N(K_1') \cap N(K_2')) \setminus (K_1' \cup K_2')$. Since (K_1, K_2) is a homogeneous pair of cliques, it follows that, for $\{i, j\} = \{1, 2\}$, N_i' is complete to K_i' and anticomplete to K_j', and Z' is complete to $K_1' \cup K_2'$. Hence, from the fact that K_1' is not anticomplete to K_2' and the fact that G is claw-free, it follows that N_1' and N_2' are cliques. Z' is anticomplete to M', because if $z \in Z'$ has a neighbor $u \in M'$, then let $a \in K_1', b \in K_2'$ be nonadjacent and observe that z is complete to the triad $\{a, b, u\}$, contrary to (5.2.2). Notice that $Y' \subseteq M'$. We start with the following claim.

(i) *Suppose that there exist $a_1, a_2 \in K_1'$, $b \in K_2'$ such that b is adjacent to a_1 and nonadjacent to a_2. Then, Z' is complete to N_1'.*

We may assume that $Z' \neq \emptyset$, because otherwise we are done. Assume for a contradiction that there exist adjacent $x_1 \in N_1', x_2 \in N_2'$ such that some $y \in Y'$ is complete to $\{x_1, x_2\}$. It follows from the fact that every vertex in Y' is simplicial that x_1 and x_2 are adjacent.

We first claim that Z' is complete to x_1. For suppose that $z \in Z'$ is nonadjacent to x_1. If z is nonadjacent to x_2, then x_1-a_2-z-b-x_2-x_1 is an induced cycle of length five, a contradiction. Therefore, z is adjacent to x_2. But now, $G|\{y, x_1, a_1, z, x_2, a_2, b\}$ is an induced heft $\mathcal{H}_3(0)$ with end $y \in Y'$, a contradiction. This proves that Z' is complete to x_1.

Now let $p \in N_1'$ and suppose that p is nonadjacent to some $z \in Z'$. Since x_1 is complete to $\{p, y, z\}$, it follows from (5.2.2) that p is adjacent to y. Since y is a simplicial vertex, and $\{p, x_2\} \in N(y)$, it follows that p is adjacent to x_2. Now, it follows from the previous argument that p is complete to Z', a contradiction. This proves **(i)**. □

We claim that Z' is a clique. For suppose that $z, z' \in Z'$ are nonadjacent. Let $x_1 \in N_1'$. Now it follows from **(i)** that x_1 is complete to the triad $\{y, z, z'\}$, contrary to (5.2.2). Thus, Z' is a clique. The last claim deals with an easy case:

(ii) *If some vertex in K_1' is complete to K_2', then the lemma holds.*

Suppose that $a_1 \in K_1'$ is complete to K_2'. First observe that no vertex in K_1' has both a neighbor and a nonneighbor in K_2', because if $a_2 \in K_1'$ has a neighbor $b_1 \in K_2'$ and a nonneighbor $b_2 \in K_2'$, then $G|\{x, x_1, x_2, a_1, a_2, b_1, b_2\}$ is an induced heft $\mathcal{H}_3(0)$ with end $y \in Y'$, a contradiction. It follows that every vertex in K_1' is either complete or anticomplete to K_2'. Since K_1' is not complete to K_2', it follows that there exists $a_2 \in K_1'$ that is anticomplete to K_2'. Now it follows from **(i)** that Z' is complete to N_1'. Thus, a_2 is a simplicial vertex and the lemma holds by (5.2.9). This proves **(ii)**. □

It follows from **(ii)** and the symmetry that we may assume that, for $\{i, j\} = \{1, 2\}$, no vertex in K_i' is complete to K_j'. Thus, it follows from **(i)** and the fact that K_1' is not complete and not anticomplete to K_2' that Z' is complete to $N_1' \cup N_2'$. We claim that $K = K_1' \cup Z' \cup N_1'$ is a dominant clique. For suppose not. Then there exists a maximal stable set S in G such that $S \cap K = \emptyset$. Let $a \in K_1'$. Since $N(a) \subseteq K \cup K_2'$, it follows that a has a neighbor in $S \cap K_2'$, because otherwise we may add a to S and obtain a larger stable set. In particular, $S \cap K_2' \neq \emptyset$ and, since K_2' is a clique, $|S \cap K_2'| = 1$. But now, the unique vertex v in $S \cap K_2'$ is complete to K_1', a contradiction. This proves that K is a dominant clique, thus proving (5.7.8). ■

One-edge ordinary leaf-blocks

The most tedious ordinary leaf blocks that we have to deal with are the blocks B that consist of just one edge. In principle, there are 15 different types of strips that we need to deal with. Lemma (5.5.4) and (5.7.7) already ruled out seven of them. Lemmas (5.7.9) to (5.7.17) deal with the remaining eight types of strips.

(5.7.9) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η)*

at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_1$, then G is resolved.

Proof. Let (J, Z) be the strip of (H, η) at F. Write $\bar{F} = \{f_1, f_2\}$. From the symmetry, we may assume that f_1 is a cut-vertex of H. Since J is a linear interval trigraph, we may order the vertices of $V(J \setminus Z)$ as $v_1, \dots, v_n$ such that for $1 \le i < k \le j \le n$, if v_i is adjacent to v_j, then v_k is strongly adjacent to v_i and v_j. From the symmetry, we may assume that $v_1 \in \eta(F, f_1)$. Now let i be smallest such that v_n is adjacent to v_i. It follows from the definition of $v_1, \dots, v_n$ that $N(v_n) = \{v_i, v_{i+1}, \dots, v_{n-1}\}$ and $N(v_n)$ is a strong clique. Therefore, v_n is a simplicial vertex and the result follows from (5.2.9). This proves (5.7.9). ■

(5.7.10) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, X) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_2$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. For convenience, we identify the vertices of J with the vertices of the member of $\mathcal{Z}_2$ to which (J, Z) is isomorphic. It follows from (5.7.6) that J contains no weakly induced cycle of length five. Let $A, B, C, X, n, \{a_i\}, \{b_i\}, \{c_i\}$ be as in the definition of $\mathcal{Z}_2$. Let $A' = A \setminus X$, $B' = B \setminus X$, $C' = C \setminus X$. Let $\{f_1, f_2\} = \bar{F}$. We may assume that f_1 is a cut-vertex of H and, from the symmetry, that $A' = \eta(F, f_1)$. We first make the following easy observation:

(i) *There are no distinct $i, j, k \in [n]$ such that $a_i, a_j \in A'$, $b_i, b_k \in B'$ and $c_i \in C'$.*

Suppose that such a_i, a_j, b_i, b_k, c_i exist. Then, c_i-a_j-a_i-b_i-b_k-c_i is a weakly induced cycle of length five, a contradiction. □

First suppose that $|B'| = 1$. Let i be such that $b_i \in B'$. Since $N(b_i) = (C' \setminus \{c_i\}) \cup \{a_i\}$, it follows from the definition of $\mathcal{Z}_2$ that b_i is simplicial and hence G is resolved by (5.2.9). So we may assume that $|B'| \ge 2$.

(ii) *If there exists $i \in [n]$ such that $a_i \in A'$, $b_i \in B'$, $c_i \in C'$, then G is resolved.*

Without loss of generality we may assume that $i = 1$. It follows from the definition of $\mathcal{Z}_2$ that c_i is strongly anticomplete to $\{a_1, b_1\}$, and a_1, b_1 are strongly adjacent. First suppose that there exists $j \in \{2, \dots, n\}$ such that $a_j \in A'$ and $b_j \in B'$. We may assume that $j = 2$. It follows from **(i)** that $A = \{a_1, b_2\}$ and $B = \{b_1, b_2\}$. If a_2 is semiadjacent to b_2, then a_1-a_2-c_1-b_2-b_1-a_1 is a weakly induced cycle of length five, a contradiction. Thus, a_2 is strongly adjacent to b_2.

We claim that $K = \{a_2, b_2\} \cup (C' \setminus \{c_2\})$ is a dominant clique in T. Clearly, K is a strong clique. So suppose that there exists a stable set $S \subseteq V(T)$ that covers K. Since, in particular, S covers c_1. Because $\{a_1, b_1, c_2\}$ is a strong clique, it follows that $S \cap \{a_1, b_1, c_2\} = \{c_2\}$. But now, no vertex in S covers b_2, a contradiction. Thus K is a dominant clique and G is resolved by (5.2.7).

So we may assume that for no $j \in \{2, \dots, n\}$, both $a_j \in A'$ and $b_j \in B'$. By this and **(i)**, it follows from the fact that $|B'| \ge 2$ that $A' = \{a_1\}$. Now let $X_1 = (B \setminus \{b_1\}) \cup \{c_1\}$ and

$X_2 = (C \setminus \{c_1\}) \cup \{a_1, b_1\}$. Observe that X_1 and X_2 are strong cliques. Since $N(X_2) \subseteq X_1$, it follows from (5.2.8) that G is resolved. This proves **(ii)**. □

In view of **(ii)**, we may assume that there is no $i \in [n]$ such that $a_i \in A', b_i \in B', c_i \in C'$. Now let $B^* = \{b_i : i \in [n], c_i \in C'\}$. It follows that B^* is strongly anticomplete to A' and $B' \setminus B^*$ is strongly complete to C. If $B^* \neq \emptyset$, then B^* is a strong clique, $N(B^*) \subseteq (B' \setminus B^*) \cup C$, and G is resolved by (5.2.8). So we may assume that $B^* = \emptyset$. Now, $B' \cup C$ is a strong clique and $N(B' \cup C) \subseteq A'$, and G is resolved by (5.2.8). This proves (5.7.10). ■

(5.7.11) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_3$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. For convenience, we identify the vertices of J with the vertices of the member of $\mathcal{Z}_3$ to which (J, Z) is isomorphic. It follows from (5.7.6) that J contains no weakly induced cycle of length five. Let $H, h_1, \dots, h_5, Z$ be as in the definition of $\mathcal{Z}_3$. Write $\bar{F} = \{f_1, f_2\}$. From the symmetry, we may assume that f_1 is a cut-vertex of H and we may assume that $h_2h_3 \in \eta(F, f_1)$.

(i) *If some vertex in $V(H) \setminus \{h_1, h_5\}$ has degree 1, then G is resolved.*

Suppose that there exists $z \in V(H) \setminus \{h_1, h_5\}$ with $\deg_H(z) = 1$. Let y be the unique neighbor of z in H. It follows from the definition of a line trigraph that yz is a simplicial vertex in T, contrary to (5.2.9). This proves **(i)**. □

By the definition of H, every edge of H is incident with one of h_2, h_3, h_4. It follows that $N(x) \subseteq \{h_2, h_3, h_4\}$ for all $x \in V(H) \setminus \{h_2, h_3, h_4\}$. By **(i)**, we may assume that no vertex is adjacent to exactly one of h_2, h_3, h_4. Thus, we may partition $V(H) \setminus \{h_1, h_2, h_3, h_4, h_5\}$ into sets X, Y_1, Y_2, Y_3 such that X is complete to $\{h_2, h_3, h_4\}$ and, for $i \in \{2, 3, 4\}$, Y_i is anticomplete to h_i and complete to $\{h_2, h_3, h_4\} \setminus \{h_i\}$.

(ii) *If $X \neq \emptyset$, then G is resolved.*

Suppose for a contradiction that $X \neq \emptyset$. Let $x \in X$. If there exists $y_2 \in Y_2$, then y_2-h_3-h_4-x-h_2-y_2 is a cycle of length five, and thus, by the definition of a line trigraph, J contains a weakly induced cycle of length five, a contradiction. If there exists $y_3 \in Y_3$, then y_3-h_2-x-h_3-h_4-y_3 is a cycle of length five, a contradiction. If there exists $x' \in X$, $x' \neq x$, then h_2-x-h_3-x'-h_4-h_2 is a cycle of length five in H, a contradiction. From this and the symmetry, it follows that $Y_2 = Y_3 = Y_3 = \emptyset$ and $|X| = 1$. Now let $A = \{h_3h_4, h_4x\}$ and let $B = \{h_2h_3, h_2x, h_3x\}$. Now, A and B are strong cliques and $N(A) = B$. Therefore, G is resolved by (5.2.8). This proves **(ii)**. □

It follows from **(ii)** that we may assume that $X = \emptyset$. Now first suppose that $Y_3 \neq \emptyset$. If there exists $y_4 \in Y_4$, then h_2-y_4-h_3-h_4-y_3-h_2 is a cycle of length five, and hence T contains a weakly induced cycle of length five, a contradiction. Therefore, by the symmetry, $Y_2 = Y_4 = \emptyset$. Let $A = \{h_4y_3 \mid y_3 \in Y_3\} \cup \{h_3h_4\}$ and let $B = \{h_2y_3 \mid y_3 \in Y_3\} \cup \{h_2h_3\}$. Then, $N(A) \subseteq N(B)$ and A and B are strong cliques and, thus, G is resolved by (5.2.8). Thus, we may assumet that $Y_3 = \emptyset$. Now, let $A = \{h_4y_2 \mid y_2 \in Y_2\} \cup \{h_3h_4\}$ and let $B = \{h_3y \mid y \in Y_2 \cup Y_4\} \cup \{h_2h_3\}$. Then, $N(A) \subseteq N(B)$ and A and B are strong cliques and, thus, G is resolved by (5.2.8). This proves (5.7.11). ■

(5.7.12) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_6$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. It follows from the definition of $\mathcal{Z}_6$ that J is a long circular interval graph that contains a simplicial vertex $z \in Z$. We may assume that J is not a linear interval trigraph, because then the result follows from (5.7.9). It follows from (5.7.6) and the fact that $\mathcal{F}$-free that J contains no weakly induced cycle of length at least five, and in particular, J contains no semihole of length at least five. We may assume that G is not resolved.

(i) *J admits a C_4-structure $(A_1, \dots, A_4, B_1, \dots, B_4)$ and there exists $i \in [4]$, such that $z \in B_i$ and $B_j = \emptyset$ for $j \in [4] \setminus \{i\}$.*

Since a trigraph of the $\bar{C}_7$ type contains no simplicial vertex, it follows from (5.4.7) that J admits a C_4-structure $(A_1, \dots, A_4, B_1, \dots, B_4)$. It follows from properties (1) and (2) of a C_4-structure that no vertex in $A_1 \cup A_2 \cup A_3 \cup A_4$ is simplicial, and hence that $z \in B_i$ for some $i \in [4]$. It follows from property (5) that A_i is strongly complete A_{i+1}. Now suppose that $B_j \neq \emptyset$ for some $j \in [4] \setminus \{i\}$. By properties (3), (4) and (5), every vertex in B_j is simplicial in T and therefore G is resolved by (5.2.9), a contradiction. It follows that $B_j = \emptyset$ for all $j \in [4] \setminus \{i\}$. This proves **(i)**. □

Let $A_1, \dots, A_4, i$ be as in the statement of **(i)**.

(ii) *A_i is strongly complete to A_{i+3} and A_{i+2} is strongly complete to A_{i+1}.*

We first claim that there exist antiadjacent $a_{i+2} \in A_{i+2}$ and $a_{i+3} \in A_{i+3}$. For suppose not. Then, $A_{i+2} \cup A_{i+3}$ and $A_i \cup A_{i+1}$ are strong cliques and $N(A_{i+2} \cup A_{i+3}) \subseteq A_i \cup A_{i+1}$. Therefore, G is resolved by (5.2.8), a contradiction. This proves the claim.

It follows from property (6) that a_{i+2} is strongly complete to A_{i+1}, and a_{i+3} is strongly complete to A_i. Now suppose that there exist antiadjacent $a_i \in A_i$ and $a'_{i+3} \in A_{i+3}$. Then, by property (6), a'_{i+3} is strongly complete to A_{i+2}. Now, a_i-a_{i+1}-a_{i+2}-a'_{i+3}-a_{i+3}-a_i, with $a_i \in A_i$, is a weakly induced cycle of length five, a contradiction. This proves that A_i is strongly complete to A_{i+3} and therefore, by the symmetry, that A_{i+2} is strongly complete to A_{i+1}, completing the proof of **(ii)**. □

It follows from **(ii)** that (A_{i+2}, A_{i+3}) is a homogeneous pair of cliques that satisfies the assumptions of (5.7.8), and therefore G is resolved by (5.7.8). This proves (5.7.12). ■

(5.7.13) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_9$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. It follows from (5.7.6) that J contains no weakly induced cycle of length five. We may assume that G is not resolved. For convenience, we identify the vertices of J with the vertices of the member of $\mathcal{Z}_9$ to which (J, Z) is isomorphic.

Let A, B, C, D, $\{a_i\}$, $\{b_i\}$, and n be as in the definition of $\mathcal{Z}_9$. Recall that, every vertex $d \in D$ is strongly adjacent to one of a_i, b_i, $i \in [n]$, and strongly antiadjacent to the other. For $i \in [n]$, we say that two vertices $d_1, d_2 \in D$ *agree* on $a_i b_i$ if $\{d_1, d_2\}$ is strongly complete to one of a_i, b_i, and strongly anticomplete to the other. They *disagree* on $a_i b_i$ otherwise.

(i) *If $d_1, d_2 \in D$ disagree on $a_i b_i$ for some $i \in [n]$, then d_1, d_2 disagree on $a_j b_j$ for every $j \in [n]$.*

From the symmetry, we may assume that d_1, d_2 disagree on $a_1 b_1$ and d_1, d_2 agree on $a_2 b_2$. From the symmetry, we may also assume that d_1 is strongly complete to $\{a_1, a_2\}$ and strongly anticomplete to $\{b_1, b_2\}$, and d_2 is strongly complete to $\{b_1, a_2\}$ and strongly anticomplete to $\{a_1, b_2\}$. But now, d_1-a_1-b_2-b_1-d_2-d_1 is a weakly induced cycle of length five, a contradiction. This proves **(i)**. □

It follows from **(i)** that D may be partitioned into two sets D_1, D_2, such that, for $i = 1, 2$, the vertices in D_i agree on all pairs $a_j b_j$, $j \in [n]$, and whenever $d_1 \in D_1$ and $d_2 \in D_2$, then d_1, d_2 disagree on all pairs $a_j b_j$, $j \in [n]$. For $\{i, j\} = \{1, 2\}$, let $A_i \subseteq A$, $B_i \subseteq B$ be the vertices in A, B, respectively, that are strongly complete to D_i and strongly anticomplete to D_j. It follows that $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $D = D_1 \cup D_2$ and, for $i = 1, 2$, $A_i \cup B_i \cup D_i$ is a strong clique.

(ii) *A_1, A_2, B_1, B_2 are all nonempty, A_1 is strongly anticomplete to B_2, and A_2 is strongly anticomplete to B_1.*

Since, for $\{i, j\} = \{1, 2\}$, every vertex in A_i has an antineighbor in B_j and vice versa, it follows that A_i, B_j are either both empty or both nonempty. If $A_1 \cup B_2 = \emptyset$, then $C \cup A_2 \cup B_1$ is a strong clique, $N(C \cup A_2 \cup B_1) \subseteq D$, and D is a strong clique, and thus G is resolved by (5.2.8), a contradiction. Hence, by the symmetry, A_1, A_2, B_1, B_2 are all nonempty.

Since every vertex in A_1 has an antineighbor in B, it follows that A_1 is not strongly complete to B_2. Now observe that (A_1, B_2) is a homogeneous pair of cliques that satisfies the assumptions of (5.7.8). It follows from (5.7.8) and the fact that G is not resolved that A_1 is strongly anticomplete to B_2. Symmetrically, it follows that A_2 is strongly anticomplete to B_1, thus proving **(ii)**. □

Now suppose for a contradiction that $D_1 = \emptyset$. It follows that $X_1 = A_1 \cup B_1 \cup C$ is a strong clique and, since $D_1 = \emptyset$, $N(X_1) \subseteq A_2 \cup B_2 \cup D_2$, which is also a strong clique. Therefore, it follows from (5.2.8) that G is resolved. It follows from the symmetry that D_1 and D_2 are both nonempty.

(iii) *C is strongly complete to at least one of D_1, D_2.*

If $c \in C$ has antineighbors $d_1 \in D_1$, $d_2 \in D_2$, then d_1-a_1-c-b_2-d_2-d_1 with $a_1 \in A_1$, $b_2 \in B_2$ is a weakly induced cycle of length five, a contradiction. We may assume that some $c_1 \in C$ has an antineighbor $d_1 \in D_1$, and some $c_2 \in C$ has an antineighbor $d_2 \in D_2$. By the previous argument, $c_1 \neq c_2$, c_1 is strongly adjacent to d_2, and c_2 is strongly adjacent to d_1. Let $a \in A_1$ and $b \in B_2$. Then, $J|\{y, d_1, c_2, c_1, d_2, a, b\}$ contains a weakly induced heft $\mathcal{H}_3(0)$ with end $y \in Y$, contrary to (5.7.6). This proves **(iii)**. □

So we may assume that C is strongly complete to D_1. Now, let $K = A_1 \cup B_1 \cup D_1 \cup C$. We claim that K is a dominant clique in T. For suppose that there exists a stable set $S \in V(T)$ that covers K. Since, in particular, S covers B_1, it follows that $S \cap A_2 \neq \emptyset$. Since $A_2 \cup B_2 \cup D_2$ is a strong clique, it follows that $|S \cap (A_2 \cup B_2 \cup D_2)| = 1$. But this implies that S does not cover A_1, a contradiction. Thus, K is a dominant clique in T and G is resolved by (5.2.7). This proves (5.7.13). ■

In the remaining strips, we will always deal with strips that are hex-expansions of three-cliqued strips. We first prove a useful lemma on hex-expansions:

(5.7.14) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$. Suppose that the strip block of (H, η) at B is ordinary and that the strip of (H, η) at F is a trigraph that is a hex-expansion of a three-cliqued trigraph (T', A, B, C). Let V_1, V_2, V_3 be as in the definition of the hex-expansion. Then, either:*

- (a) *G is resolved, or*
- (b) *B and C are nonempty, and $V_1 \cup V_2 \cup V_3$ is a strong clique.*

Proof. Let (J, Z) be the strip of (H, η) at F. Let (T', A, B, C) be such that J is a hex-expansion of (T', A, B, C) with $z \in A$, and let V_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.*, V_1 is strongly complete to $B \cup C$, V_2 is strongly complete to $A \cup C$, and V_3 is strongly complete to $A \cup B$. It follows from (5.7.6) that J contains no weakly induced cycle of length five. We may assume that G is not resolved, because otherwise outcome (a) holds.

First suppose that $V_1 \cup V_2 \cup V_3$ is a strong clique. If B is empty, then $V_1 \cup C \cup V_2$ is a strong clique that only has neighbors in the strong clique $A \cup V_2 \cup V_3$, and hence G is resolved by (5.2.8). Thus, by the symmetry, B and C are both nonempty, and hence outcome (b) holds. So we may assume that $V_1 \cup V_2 \cup V_3$ is not a strong clique.

Next, if B is strongly complete to C, then $B \cup C \cup V_1$ is a strong clique that only has neighbors in the strong clique $A_1 \cup V_2 \cup V_3$, and hence G is resolved by (5.2.8), a contradiction. It follows that B is not strongly complete to C,

(i) *V_1 is strongly complete to one of V_2, V_3.*

Since B is not strongly complete to C, there exist antiadjacent $b \in B$, $c \in C$. First suppose that $v_1 \in V_1$ has antineighbors $v_2 \in V_2$ and $v_3 \in V_3$. Then, v_2-c-v_1-b-v_3-v_2 is a weakly induced cycle of length five, a contradiction. This proves that no vertex in V_1 has an antineighbor in both V_2 and V_3. So we may assume that there exist antiadjacent $v_1 \in V_1, v_2 \in V_2$ and antiadjacent $v_1' \in V_2$ and $v_3 \in V_3$. It follows that v_1 is strongly adjacent to v_3 and v_1' is strongly adjacent to v_2. Now, $J|\{z, v_1, v_1', v_2, v_3, b, c\}$ contains a weakly induced heft $\mathcal{H}_3(0)$ with end $z \in Z$, a contradiction. This proves **(i)**. □

In view of **(i)** and the symmetry, we may assume that V_1 is strongly complete to V_2 and V_1 is not strongly complete to V_3. Let $C' \subseteq C$ be all vertices in C that have a neighbor in A.

(ii) *C' is strongly complete to B.*

Suppose that $c \in C'$ has an antineighbor $b \in B$. Since $c \in C'$, c has a neighbor $a \in A$. Because a is not complete to the triad $\{b, c, z\}$, it follows that a is strongly antiadjacent to b. Now, v_3-a-c-v_1-b-v_3, with $v_1 \in V_1$ and $v_3 \in V_3$ antiadjacent, is a weakly induced cycle of length five, a contradiction. This proves **(ii)**. □

Since B is not strongly complete to C, it follows that $C \setminus C' \neq \emptyset$. If $V_2 = \emptyset$, then $C \setminus C'$ is a strong clique, $N(C \setminus C') \subseteq B \cup C' \cup V_1$ is a strong clique, and thus G is resolved by (5.2.8), a contradiction. Therefore, $V_2 \neq \emptyset$.

(iii) *There are no $a \in A$, $b, b' \in B$, $c \in C$ such that both a and c are mixed on $\{b, b'\}$.*

Suppose that such a, b, b', c' exist. From the symmetry, we may assume that a is adjacent to b and antiadjacent to b'. Because C' is strongly complete to B, it follows that $c \in C \setminus C'$. Thus, c is strongly antiadjacent to a. If c is adjacent to b and antiadjacent to c', then b is complete to the triad $\{a, b', c\}$, contrary to (5.2.2). Thus, c is adjacent to b' and antiadjacent to b. Let $v_2 \in V_2$. Now, v_2-a-b-b'-c-v_2 is a weakly induced cycle of length five, a contradiction. This proves **(iii)**. □

Since B is not strongly complete to C, it follows from **(ii)** that some $b \in B$ and $c \in C \setminus C'$ are antiadjacent. If b and c are semiadjacent, then G is resolved by (5.7.8) applied to b-c-v_2-v_1-b with $v_1 \in V_1$ and $v_2 \in V_2$, a contradiction. Thus, $c \in C$ is not semiadjacent to any vertex in B. If c is strongly anticomplete to B, then c is simplicial and thus G is resolved by (5.2.9), a contradiction. Hence, c has a strong neighbor $b' \in B$. Therefore, c is mixed on $\{b, b'\}$ and hence, by **(iii)**, no vertex in A is mixed on $\{b, b'\}$. Since every vertex in $B \setminus \{b, b'\}$ is adjacent to one of b, b', it follows that no vertex in A is mixed on B. Now, (B, C) is a homogeneous pair of cliques that satisfies (5.7.8) and hence G is resolved by (5.7.8), a contradiction. This proves (5.7.14). ■

(5.7.15) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_{10}$, then G is resolved.*

Proof. Let $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, c_1, c_2, A, B, C, X$ be as in the definition of $\mathcal{Z}_{10}$. Let V_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.*, V_1 is strongly complete to $B \cup C$, V_2 is strongly complete to $A \cup C$, and V_3 is strongly complete to $A \cup B$. It follows from (5.7.14) that we may assume that $V_1 \cup V_2 \cup V_3$ is a strong clique. We first note that if $\{b_2, b_3\} \subseteq X$, then $N(c_1) = V_1 \cup V_2 \cup \{c_2\}$, and hence c_1 is a simplicial vertex. Therefore, by (5.2.9), we may assume that at least one of b_2, b_3 is not in X. It follows from the fact that either $a_2 \in X$ or $\{b_2, b_3\} \subseteq X$, that $a_2 \in X$. If $d \in X$, then it follows that $N(b_0) = \{b_1, b_2, b_3\} \cup V_1 \cup V_3$ and hence b_0 is a simplicial vertex. Thus, by (5.2.9), we may assume that $d \notin X$. Now, if $b_2 \notin X$, then b_0-d-a_1-c_2-c_1-b_2-b_0 is a weakly induced cycle of length six, a contradiction. Therefore, $b_2 \in X$ and $b_3 \notin X$. Now, b_0-d-a_1-c_2-c_1-b_3-b_0 is a weakly induced cycle of length six, a contradiction. This proves (5.7.15). ■

(5.7.16) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_{11}$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. It follows from (5.7.6) that J contains no weakly induced cycle of length five. Let a_0, b_0, A, B, C, X be as in the definition of $\mathcal{Z}_{11}$ Let $A' = A \setminus (X \cup \{a_0\})$, $B' = B \setminus (X \cup \{b_0\})$, $C' = C \setminus X$. Let V_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.*, V_1 is strongly complete to $B' \cup C'$, V_2 is strongly complete to $A' \cup C'$, and V_3 is strongly complete to $A' \cup B'$.

It follows from (5.7.14) that we may assume that $V_1 \cup V_2 \cup V_3$ is a strong clique, and B' is nonempty. If $a_0 \in X$ or a_0 is strongly antiadjacent to b_0, then $N(b_0) = B' \cup V_1 \cup V_3$ and hence b_0 is a simplicial vertex, and G is resolved by (5.2.9). So we may assume that $a_0 \notin X$ and a_0 is semiadjacent to b_0.

We claim that $N(C)$ is a strong clique. For suppose not. Then there exist antiadjacent $u_1, u_2 \in N(C)$. Since $N(C) \subseteq A' \cup B' \cup V_1 \cup V_2$, B' is strongly complete to V_1, and A' is strongly complete to V_2, we may assume that $u_1 \in A' \cup V_2$ and $u_2 \in B' \cup V_1$. Because $u_1, u_2 \in N(C)$, there exists a weakly induced path P from u_1 to v_2 such that $V(P^*) \subseteq C$ and $|V(P)| \in \{3, 4\}$. Now, a_0-u_1-P-u_2-b_0-a_0 is a weakly induced cycle of length five or six, a contradiction. ■

(5.7.17) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_{15}$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. It follows from (5.7.6) that J contains no weakly induced cycle of length five. Let A, B, C, X be as in the definition of $\mathcal{Z}_{15}$. Let V_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.*, V_1 is strongly complete to $B \cup C$, V_2 is strongly complete to $A \cup C$, and V_3 is strongly complete to $A \cup B$. It follows from (5.7.14) that we may assume that $V_1 \cup V_2 \cup V_3$ is a strong clique. If v_2 is semiadjacent to v_5, then $(\{v_2\}, \{v_5\})$ form a homogeneous pair of cliques in T that satisfy the assumptions of (5.7.8) and thus G is resolved by (5.7.8). Therefore, we may assume that v_2 is strongly antiadjacent to v_5. Moreover, if $X = \emptyset$, then $J|\{v_1, v_2, \dots, v_8\}$ contains a weakly induced heft $\mathcal{H}_3(1)$, a contradiction. From the symmetry, we may assume that $v_4 \in X$. But now, $N(v_2) \subseteq \{v_1, v_3\} \cup V_2 \cup V_3$ is a strong clique. Thus, v_2 is a simplicial vertex and, hence, G is resolved by (5.2.9). This proves (5.7.17). ■

Multi-edge ordinary leaf-blocks

The previous subsection dealt with ordinary leaf-blocks that consist of exactly one edge. The following lemmas deal with the remaining cases when an ordinary leaf-block consists of multiple edges. Recall from (5.7.6) that such a leaf-block B is of the $\mathcal{B}_2$ type, and hence $U(B)$ is isomorphic to one of K_2, K_3, K_4, $K_{2,t}$, or $K^+_{2,t}$ $(t \geq 2)$. We start with the case K_2:

(5.7.18) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H such that $U(B)$ is isomorphic to K_2, and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. Then, G is resolved.*

Proof. We may assume that G is not resolved. Let $\{u, v\} = V(B)$ such that u is the unique cut-vertex of H that belongs to $V(B)$. It follows from (5.7.6) that either $\ell(F) \in \{\{1\}, \{2\}\}$ for all $F \in E(B)$, or there exists $F^* \in E(B)$ with $\ell(F^*) \subseteq \{2, 3\}$ and $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$.

First suppose that $\ell(F) \in \{\{1\}, \{2\}\}$ for all $F \in E(B)$. Since G is not resolved, it follows from (5.7.3) that $\eta(F) = \eta(F, u) \cup \eta(F, v)$ for all $F \in E(H)$ with $\ell(F) = \{2\}$. But now let

$$M_1 = \bigcup\{\eta(F, u) \mid F \in E(H)\} \text{ and } M_2 = \bigcup\{\eta(F, v) \mid F \in E(H), \ell(F) = \{2\}\}.$$

It follows from the definition of a strip-structure that M_1 and M_2 are strong cliques and $N(M_2) \subseteq N(M_1)$. Hence, the lemma holds by (5.2.8).

So we may assume that there exists $F^* \in E(B)$ with $3 \in \ell(F^*)$ and $\ell(F) = \{1\}$ for all $F \in E(H)\setminus\{F^*\}$. It follows from (5.5.7) that $\ell(F^*) \neq \{3\}$ and hence $\ell(F^*) = \{2, 3\}$. Let $A = \eta(F^*, u)$, $B = \eta(F^*, v)$, and $C = \eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$. It follows from (5.5.10) that C is a strong clique. Let (J, Z) be the strip of (H, η) at F^*, let z_1 be the unique vertex in Z that is strongly complete to A and let z_2 be the unique vertex in Z that is strongly complete to B. Let $M = \bigcup\{\eta(F) \mid F \in E(H) \setminus \{F^*\}\}$. It follows that M is strongly complete to $A \cup B$ and strongly anticomplete to C.

(i) *At least one of A, B is not mixed on C.*

Suppose that A and B are both mixed on C. Construct the trigraph T' from T by making A strongly anticomplete to B. It follows from (5.5.6) applied to T' that there exists a weakly induced path $P = p_1\text{-}p_2\text{-}p_3\text{-}p_4$ in T' with $p_1 \in A$, $p_2, p_3 \in C$, and $p_4 \in B$. If p_1, p_4 are antiadjacent in T, then $x\text{-}P\text{-}x$ with $x \in M$ is a weakly induced cycle of length five in D, a contradiction. Thus, p_1, p_4 are adjacent. But now, p_1 is complete to the triad $\{z_1, p_2, p_4\}$, contrary to (5.2.2). This proves **(i)**. □

Now let $A' \subseteq A$, $B' \subseteq B$ be the vertices in A, B, respectively, that have a neighbor in C. It follows that $A \setminus A'$ is strongly anticomplete to C and, because J is claw-free, to B'. It follows that $B \setminus B'$ is strongly anticomplete to C and, because J is claw-free, to A'.

(ii) *$A = A'$ and $B = B'$.*

If $N(C)$ is a strong clique, then it follows from (5.2.8) that G is resolved, a contradiction. Thus, there exists $c \in C$ that has antiadjacent neighbors $a \in A'$ and $b \in B'$. We claim that $A \setminus A'$ is strongly anticomplete to $B \setminus B'$. For suppose that there exist adjacent $a \in A \setminus A'$ and $b \in B \setminus B'$. Then, a-a'-c-b'-b-a is a weakly induced cycle of length five, a contradiction. Now suppose that one of $A \setminus A'$, $B' \setminus B'$ is nonempty. Then, because $N[A \setminus A'] \subseteq \bigcup\{\eta(F,u) \mid F \in E(H), u \in \bar{F}\}$ and $N[B \setminus B'] = M$ are strong cliques, it follows from (5.2.8) that G is resolved, a contradiction. Therefore, $A = A'$ and $B = B'$. □

By **(i)**, at most one of A, B is mixed on C. Since every vertex in $A \cup B$ has a neighbor in C, it follows that at least one of A, B is strongly complete to C. If B is strongly complete to C, then, because $N[B \cup C] \subseteq M \cup A$, (5.2.8) implies that G is resolved. Thus, we may assume that A is strongly complete to C and B is not strongly complete to C. Let $B'' \subseteq B$ be the set of vertices in B that are not strongly complete to C. It follows from our assumptions that $B'' \neq \emptyset$. Since J is claw-free, it follows that B'' is strongly anticomplete to A. Now, (B'', C) is a homogeneous pair of cliques that satisfies the assumptions of (5.7.8) and, thus, G is resolved by (5.7.8). This proves (5.2.9). ■

This leaves the cases K_3, K_4, $K_{2,t}$ and $K_{2,t}^+$, all of which we deal with in the next lemma:

(5.7.19) *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H such that $U(B)$ is isomorphic to K_3, K_4, $K_{2,t}$, or $K_{2,t}^+$ for some $t \geq 2$, and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. Then, G is resolved.*

Proof. Let $V(B) = \{v_1, \dots, v_k\}$ with $k = |V(B)|$. We may assume that v_1 is the unique cutvertex of H in $V(B)$.

(i) *If $U(B)$ is isomorphic to K_3, then G is resolved.*

From (5.7.6), it follows that $z \leq 2$ for all $z \in \ell(F)$ with $F \in E(B)$. First suppose that $\ell(F) = \{1\}$ for all $F \in E(H)$ with $\bar{F} = \{v_2, v_3\}$. Then,

$$\bigcup\{\eta(F) \mid F \in E(H), \bar{F} = \{v_2, v_3\}\}$$

is a strong clique and all its neighbors are in the strong clique

$$\bigcup\{\eta(F, v_1) \mid F \in E(H), v_1 \in \bar{F}\}.$$

Thus, the lemma holds by (5.2.8). So we may assume that there exists $F^* \in E(H)$ with $\bar{F}^* = \{v_2, v_3\}$ and $\ell(F^*) = \{2\}$. We may also assume that G is not resolved. It follows from (5.7.3) that $\eta(F^*) = \eta(F^*, v_2) \cup \eta(F^*, v_3)$. But now, $(\eta(F^*, v_2), \eta(F^*, v_3))$ is a homogeneous pair of cliques that satisfies the assumptions of (5.7.8), and thus G is resolved by (5.7.8). □

(ii) *If $U(B)$ is isomorphic to K_4, then G is resolved.*

Because every edge in $E(B)$ is in a cycle of length four in B, it follows from (5.5.1) that $\ell(F) = \{1\}$ for all $F \in E(H)$. Now,

$$\bigcup \{\eta(F) \mid F \in E(H), \bar{F} \subseteq \{v_2, v_3, v_4\}\}$$

is a strong clique and all its neighbors are in the strong clique

$$\bigcup \{\eta(F, v_1) \mid F \in E(H), v_1 \in \bar{F}\}.$$

Thus, G is resolved by (5.2.8). □

(iii) *If $U(B)$ is isomorphic to $K_{2,t}$ or $K_{2,t}^+$ for some $t \geq 2$, then G is resolved.*

Observe that every edge in $E(B)$ is in a cycle of length four in B. Therefore, it follows from (5.5.1) that $\ell(F) = \{1\}$ for all $F \in E(H)$. Let $V(B) = X \cup Y$ such that X is a stable set of size t and $|Y| = 2$. Let v be the unique cutvertex of H in B. If $v \in Y$, then $p, p' \in X$ satisfy the assumptions of (5.7.2), and hence that G is resolved. Thus, we may assume that $v \in X$. First assume that $U(B)$ is not isomorphic to $K_{2,2}^+$. If $U(B)$ is isomorphic to $K_{2,t}$, then let $p, p' \in Y$. Otherwise, $t \geq 3$, and let $p, p' \in X \setminus \{v\}$. Now, p and p' satisfy the assumptions of (5.7.2), and hence G is resolved.

So we may assume that $U(B)$ is isomorphic to $K_{2,2}^+$. We may also assume that G is not resolved. Let $Y = \{y_1, y_2\}$. It follows from (5.7.6) that $\ell(F) \subseteq \{1, 2\}$ for every $F \in E(H)$ with $\bar{F} = \{y_1, y_2\}$. Moreover, it follows from (5.5.10) that $\eta(F) = \eta(F, y_1) \cup \eta(F, y_2)$ for every $F \in E(H)$ with $\bar{F} = \{y_1, y_2\}$ and $\ell(F) = \{2\}$. Now, let

$$Z_1 = \bigcup \{\eta(F, y_1) \mid F \in E(H), y_1 \in F\}$$

and

$$Z_2 = \bigcup \{\eta(F, y_2) \mid F \in E(H), y_2 \in \bar{F}, \ell(F) = \{2\}\}.$$

It follows that Z_1 and Z_2 are strong cliques and $N(Z_2) \subseteq N(Z_1)$. Thus, G is resolved by (5.2.8). □

This proves (5.7.19). ■

5.7.4 Proof of (5.7.1)

We are finally in a position to prove (5.7.1):

(5.7.1). *Every connected $\mathcal{F}$-free nonbasic claw-free graph is resolved.*

Proof. Let G be a connected $\mathcal{F}$-free nonbasic claw-free graph. It follows from (5.2.6) that G is a graphic thickening of some claw-free trigraph that admits a proper strip-structure. Therefore, by

(5.5.1), G has an optimal representation (T, H, η). It follows from (5.5.2) that, for each strip (J, Z), either

(a) (J, Z) is a spot, or

(b) (J, Z) is a isomorphic to a member of $\mathcal{Z}_0$.

If H is 2-connected, then it follows from (5.7.4) that G is resolved. Thus, we may assume that H is not 2-connected. Therefore, let $(B_1, B_2, \dots, B_q)$, with $q \geq 2$, be the block-decomposition of H. Since $q \geq 2$, H has at least two leaf-blocks B, B'. It follows from (5.7.5) that the strip-block of (H, η) at at least one of these two blocks, B say, is ordinary with respect to G.

First suppose that $|E(B)| = 1$. Let $F \in E(B)$. It follows from (5.5.4) and (5.7.7) that the strip (J, Z) of (H, η) at F is either a spot or is isomorphic to a member of one of $\mathcal{Z}_1$, $\mathcal{Z}_2$, $\mathcal{Z}_3$, $\mathcal{Z}_6$, $\mathcal{Z}_9$, $\mathcal{Z}_{10}$, $\mathcal{Z}_{11}$, or $\mathcal{Z}_{15}$. If (J, Z) is a spot, then the unique vertex in $V(J) \setminus Z$ is a simplicial vertex and the result follows from (5.2.9). Thus, we may assume that (J, Z) is not a spot. Now, the theorem follows from (5.7.9), (5.7.10), (5.7.11), (5.7.12), (5.7.13), (5.7.15), (5.7.16), (5.7.17), respectively. So we may assume that $|E(B)| \geq 2$. It follows from (5.7.6) that there exists $t \geq 2$ such that $U(B)$ is isomorphic to one of K_2, K_3 K_4, $K_{2,t}$, or $K_{2,t}^+$. Thus, the theorem follows from (5.7.18) and (5.7.19). This proves (5.7.1). ■

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